

LIFTING TORSION GALOIS REPRESENTATIONS

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ABSTRACT. Let p be an odd prime and \mathcal{O}/\mathbb{Z}_p be of degree $d = ef$ with uniformiser π and residue field \mathbb{F}_{p^f} . Let $\rho_n : G_{\mathbb{Q}} \rightarrow GL_2(\mathcal{O}/(\pi^n))$ have modular mod π reduction $\bar{\rho}$, be ordinary at p , and satisfy some mild technical conditions. We show that ρ_n can be lifted to a characteristic 0 geometric representation which arises from a newform. Earlier methods could not handle the case of $n > 1$ when $e > 1$.

We also show that a prescribed totally ramified complete discrete valuation ring \mathcal{O} is the weight 2 deformation ring for $\bar{\rho}$ (or the semistable or flat quotient of this ring in a few cases) for a suitable choice of auxiliary level. This implies the field of Fourier coefficients of newforms of square free level that give rise to $\bar{\rho}$ can have arbitrarily large ramification index at p .

In an appendix, we use some of the work in the paper to prove by p -adic approximation, the modularity of many ordinary, geometric, finitely ramified representations $\rho : G_{\mathbb{Q}} \rightarrow GL_2(\mathcal{O})$ assuming the reduction is modular and irreducible. The previous methods along these lines in [7] applied only for \mathcal{O}/\mathbb{Z}_p unramified. In the appendix we are recovering known results of Wiles and Taylor-Wiles by a different method.

1. INTRODUCTION

Let $\bar{\rho} : G_{\mathbb{Q}} \rightarrow GL_2(k)$ be a continuous, odd, irreducible representation of the absolute Galois group $G_{\mathbb{Q}}$ of \mathbb{Q} with k a finite field of characteristic p . We call such a representation of S -type.

We assume p is odd. The second author introduced in [13] a method of killing dual Selmer groups to show that there were geometric lifts (i.e. ramified at only finitely many primes, and deRham at p) of (most) $\bar{\rho}$ of S -type, with ramification allowed at a set of auxiliary primes Q . (Also see [18] where the language of dual Selmer groups was introduced.) The goal of that work was to show that $\bar{\rho}$ did have in the first place a characteristic zero lift without assuming that $\bar{\rho}$ was modular. Let $W(k)$ be the ring of Witt vectors. Then the same method applied to lifting representations $\rho_n : G_{\mathbb{Q}} \rightarrow GL_2(W(k)/p^n)$, with reduction $\bar{\rho}$ of S -type, to geometric characteristic 0 representations if $p > 3$ provided ρ_n is balanced in a sense made precise below.

In this paper we address the question of producing geometric lifts of representations $\rho_n : G_{\mathbb{Q}} \rightarrow GL_2(\mathcal{O}/(\pi^n))$ with reduction $\bar{\rho}$ where \mathcal{O} is the ring of integers of a ramified extension of \mathbb{Q}_p . The method of [13] does not work directly as when \mathcal{O}/\mathbb{Z}_p ramified, the map $GL_2(\mathcal{O}/(\pi^2)) \rightarrow GL_2(\mathcal{O}/(\pi))$ is split. We do this under the conditions that ρ_n is balanced, ordinary and has full image (cf. §2).

There are two steps. In Theorem 11 we firstly show that ρ_n lifts to a characteristic 0, ordinary representation which is finitely ramified. Using this, in Theorem 20, for which we only consider the case when \mathcal{O} is totally ramified, we show in fact that we can also get lifts with the further property of being geometric at p .

For this purpose, we first study the ordinary (no weight is fixed) deformation rings $R^{ord, Q-new}$, and construct sets of primes Q such that $R^{ord, Q-new}$ has 1-dimensional tangent space and thus is a quotient of $W(\mathbb{F}_{p^f})[[U]]$. Further ρ_n arises as a specialization of the universal representation associated to $R^{ord, Q-new}$. As a certain dual Selmer group vanishes, we in fact deduce that

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$R^{ord, Q-new} \simeq W(\mathbb{F}_{p^f})[[U]]$. This already ensures that ρ_n lifts to an ordinary characteristic 0 representation which is ramified at finitely many primes (cf. Theorem 11). We also deduce that $R^{ord, Q-new}$ is isomorphic to the related Hida Hecke algebra $\mathbb{T}^{ord, Q-new}$, as the latter is known to be non-zero and flat over $\mathbb{Z}_p[[T]]$ by level raising results of [4] and Hida's results on the structure of ordinary Hecke algebras respectively.

In Theorem 20 we further show that by a more careful choice of Q , the weight 2 quotient of $R^{ord, Q-new}$ (or more precisely the semistable or flat quotient of this weight 2 ring in a few cases) can be controlled to be \mathcal{O} and ρ_n arises from this weight 2 quotient. The proof of Theorem 20 is more involved than the proof of Theorem 11. Essential use is made of the techniques of [8].

In [13] no assumption of modularity on $\bar{\rho}$ was made. In proving Theorem 20, we only use modularity of $\bar{\rho}$ in one place (cf. Lemma 21). One of the interests of this theorem is that we manage to lift the representation ρ_n to a characteristic 0 geometric lift (of weight 2), in spite of it being impossible to kill the minimal weight 2 dual Selmer (and Selmer) group of $\bar{\rho}$ using primes which are nice for ρ_n (not just $\bar{\rho}$!) when \mathcal{O}/\mathbb{Z}_p is ramified.

This brings to mind a folklore question:

Question 1. *Let K be an imaginary quadratic field and let $\bar{\rho} : G_K \rightarrow GL_2(k)$ be irreducible. Do there always exist characteristic 0 geometric lifts of $\bar{\rho}$?*

Indeed its hard to guess an answer either way with any confidence! Here too a principal obstacle to lifting is that a certain dual Selmer group cannot be killed. Presently we do not see if our methods will help in answering the question affirmatively.

In the appendix, we use the isomorphisms $W(\mathbb{F}_{p^f})[[U]] \simeq R^{ord, Q-new} \simeq \mathbb{T}^{ord, Q-new}$ above to extend the approach of [7] to modularity of geometric lifts ρ of $\bar{\rho}$ by p -adic approximation, without imposing the condition that ρ is defined over the Witt vectors. This condition was essential in [7] again because $GL_2(\mathcal{O}/(\pi^2)) \rightarrow GL_2(\mathcal{O}/(\pi))$ is split if \mathcal{O}/\mathbb{Z}_p is ramified. For a similar reason it was assumed in [7] that $p > 3$ (as the homomorphism $GL_2(\mathbb{Z}/9) \rightarrow GL_2(\mathbb{Z}/3)$ is split), an assumption that we remove in the appendix. The appendix uses only Theorem 11 of this paper (and not the more involved Theorem 20), together with level lowering arguments. We also note that Jack Thorne in [19] has used the strategy of [7] to prove modularity lifting theorems in new cases.

In the paper we have not striven for generality, and have opted to present the new ideas of this paper in their simplest setting.

Acknowledgements: Much of §2, §3 and Theorem 19 of this paper is unpublished work of Benjamin Lundell in his Cornell Ph.D. thesis, [9]. The results presented here are slightly more general. The authors thanks G. Böckle, B. Lundell and M. Stillman for helpful conversations.

2. THE SETUP

Let $p \geq 3$ be a prime and \mathcal{O} be the ring of integers of a finite extension of \mathbb{Q}_p with uniformiser π and residue field $k = \mathbb{F}_{p^f}$, a finite extension of \mathbb{F}_p . Suppose $[\mathcal{O} : \mathbb{Z}_p] = d = ef$ with e, f the ramification and inertial degrees. Let ϵ be the p -adic cyclotomic character and let $\bar{\epsilon}$ be its mod p reduction.

We consider representations $\bar{\rho} : G_{\mathbb{Q}} \rightarrow GL_2(k)$ of S -type with k of odd characteristic and lifts $\rho_n : G_{\mathbb{Q}} \rightarrow GL_2(\mathcal{O}/(\pi^n))$ and $\rho : G_{\mathbb{Q}} \rightarrow GL_2(\mathcal{O})$ of $\bar{\rho}$. We assume that these representations are odd, ordinary at p , have full image (i.e. the image contains $SL_2(k)$, $SL_2(\mathcal{O}/(\pi^n))$ and $SL_2(\mathcal{O})$ respectively) and have determinant ϵ , that $\bar{\rho}$ is modular and that all these representations are

of weight 2, by which we mean that their restriction to an inertia group I_p at p is conjugate to $\begin{pmatrix} \epsilon & * \\ 0 & 1 \end{pmatrix}$, with ϵ the cyclotomic character.¹

At p we require that

- If $\bar{\rho}|_{G_p} = \begin{pmatrix} \bar{\epsilon} & * \\ 0 & 1 \end{pmatrix}$ is flat then $\rho_n|_{G_p}$ is either flat or semistable, that is if $\rho_n|_{G_p}$ is not flat, $\rho_n|_{G_p} = \begin{pmatrix} \epsilon & * \\ 0 & 1 \end{pmatrix}$ and the $*$ arises from taking a p -power root of a nonunit of \mathbb{Z}_p .
- If $\bar{\rho}|_{G_p} = \begin{pmatrix} \bar{\epsilon} & 0 \\ 0 & 1 \end{pmatrix}$ then $\rho_n|_{G_p}$ is flat.

The first condition on $\rho_n|_{G_p}$ is simply requiring that it lift to characteristic 0 exist and is therefore necessary. The second condition is more restrictive. We do not know how to deal with the second case above if $\rho_n|_{G_p}$ is not flat.

Let S_0 and S be the sets of ramified primes of $\bar{\rho}$ and ρ_n respectively (these include p).

We assume $\bar{\rho}$ is ordinary and modular of weight 2 (hence odd) and trivial nebentype so $S \supset S_0 \supset \{p, \infty\}$. We also assume that ρ_n is **balanced** at weight 2. To understand this last condition, which involves properties of ρ_n at $v \neq p$, we recall some basics on Selmer and dual Selmer groups.

For each $v \in S_0$ a smooth quotient of the versal weight 2 deformation ring of $\bar{\rho}|_{G_v}$ has been defined on pages 120-124 of [13] and in [18]. The points of (the *Spec* of) this smooth quotient are our allowable deformations and are denoted \mathcal{C}_v . Corresponding to the tangent space of this smooth quotient is a subspace $\mathcal{L}_v \subset H^1(G_v, \text{Ad}^0 \bar{\rho})$. Fact 1 follows from the discussion in [13] referred to above.

Fact 1. *For all $v \in S_0$ there exist \mathcal{C}_v and \mathcal{L}_v as above satisfying $\dim \mathcal{L}_v = \dim H^0(G_v, \text{Ad}^0 \bar{\rho}) + \delta_{vp}$, where $\delta_{vp} = 0$ or 1 depending on $v \neq p$ and $v = p$.*

Let M be an $\mathbb{F}_{p^f}[G_{\mathbb{Q}}]$ -module with \mathbb{G}_m -dual M^* and let R be the union of the places whose inertial action on M is nontrivial and $\{p, \infty\}$. Let $\mathcal{M}_v \subset H^1(G_v, M)$ with annihilator $\mathcal{M}_v^\perp \subset H^1(G_v, M^*)$ under the perfect local pairing

$$H^1(G_v, M) \times H^1(G_v, M^*) \rightarrow H^2(G_v, \mathbb{F}_{p^f}(1)) \simeq \mathbb{F}_{p^f}.$$

Set the Selmer group for the subspaces $\mathcal{M}_v \subset H^1(G_v, M)$ to be

$$H_{\mathcal{M}}^1(G_R, M) := \text{Ker} \left(H^1(G_R, M) \rightarrow \bigoplus_{v \in R} \frac{H^1(G_v, M)}{\mathcal{M}_v} \right)$$

and the dual Selmer group

$$H_{\mathcal{M}^\perp}^1(G_R, M^*) := \text{Ker} \left(H^1(G_R, M^*) \rightarrow \bigoplus_{v \in R} \frac{H^1(G_v, M^*)}{\mathcal{M}_v^\perp} \right)$$

Recall Proposition 1.6 of [20]:

Proposition 2.

$$\begin{aligned} & \dim H_{\mathcal{M}}^1(G_R, M) - \dim H_{\mathcal{M}^\perp}^1(G_R, M^*) \\ &= \dim H^0(G_{\mathbb{Q}}, M) - \dim H^0(G_{\mathbb{Q}}, M^*) + \sum_{v \in R} (\dim \mathcal{M}_v - \dim H^0(G_v, M)). \end{aligned}$$

¹This assumption is mainly for convenience and we could have gotten by assuming that the representations restricted to I_p are of the form $\begin{pmatrix} \epsilon^{k-1} & * \\ 0 & 1 \end{pmatrix}$ with $k \geq 2$ and $\bar{\rho}$ is distinguished at p , with the further conditions on ρ_n below when $k = 2$ and $\bar{\rho}$ is split at p . The assumption on the determinant is also of a similar nature.

Fact 1, Proposition 2 and the fact that $\bar{\rho}$ is odd with full image together imply the ordinary weight 2 Selmer group $H_{\mathcal{L}}^1(G_{S_0}, Ad^0 \bar{\rho})$ and its dual Selmer group $H_{\mathcal{L}^\perp}^1(G_{S_0}, Ad^0 \bar{\rho})$ have the same dimension. We need the following

Balancedness Assumption. Let $v \in S \setminus S_0$. We assume a smooth quotient of the versal deformation ring for $\bar{\rho}|_{G_v}$ exists with points \mathcal{C}_v and induced subspace $\mathcal{L}_v \subset H^1(G_v, Ad^0 \bar{\rho})$ such that $\dim H^0(G_v, Ad^0 \bar{\rho}) = \dim \mathcal{L}_v$.

The local conditions \mathcal{L}_v for $v \in S_0$ are now standard so we do not recall their definitions. See [13] and [18]. We will be doing ordinary at p deformation theory with arbitrary weights so we will also work with $Ad\bar{\rho}$, the full adjoint, as well. We define local conditions $\tilde{\mathcal{L}}_v$ for the full adjoint.

Definition 3. 1) For $v \neq p$ set

$$\tilde{\mathcal{L}}_v := \mathcal{L}_v \oplus H_{nr}^1(G_v, \mathbb{F}_{p^f}) \subset H^1(G_v, Ad^0 \bar{\rho}) \oplus H^1(G_v, \mathbb{F}_{p^f}) = H^1(G_v, Ad\bar{\rho}),$$

that is the direct sum of \mathcal{L}_v and the \mathbb{F}_{p^f} -valued unramified twists in the dual numbers. Set $\tilde{\mathcal{C}}_v$ to be all unramified twists of the points \mathcal{C}_v .

2) For $v = p$, set $W = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}$ and $\tilde{\mathcal{L}}_p = \text{Ker}(H^1(G_p, Ad\bar{\rho}) \rightarrow H^1(I_p, Ad\bar{\rho}/W))$. Then $\tilde{\mathcal{L}}_p \supset \mathcal{L}_p$ and $\tilde{\mathcal{L}}_p \supset H_{nr}^1(G_p, \mathbb{F}_{p^f})$, the unramified twists, though it need not be the direct sum of these subspaces. Set $\tilde{\mathcal{C}}_p$ to be the ordinary deformations of $\bar{\rho}$ of any weight.

3. LOCAL DEFORMATION RINGS

3.1. Ordinary deformation rings at p . We need Lemmas 4 and 5 for 5) of Proposition 6.

Lemma 4. Let L be the composite of $\mathbb{Q}_p(\mu_{p^\infty})$ and the \mathbb{Z}_p -unramified extension of \mathbb{Q}_p . Let $\psi : G_p \rightarrow \mathbb{Z}_p^*$ be a character. Then there exists a nonsplit representation $\rho_\psi : G_p \rightarrow GL_2(\mathbb{Z}_p)$ with $\rho_\psi = \begin{pmatrix} \psi & * \\ 0 & 1 \end{pmatrix}$ where $* \equiv 0 \pmod{p}$ but $* \not\equiv 0 \pmod{p^2}$. Furthermore, after base change to L , the $*$ arises, via Kummer theory, by taking p -power roots of units of elements of L .

Proof. That ρ_ψ exists follows from the well-known fact that

$$\dim H^1(G_p, \mathbb{Q}_p(\psi)) = \begin{cases} 2 & \psi = \epsilon \\ 1 & \psi \neq \epsilon \end{cases}.$$

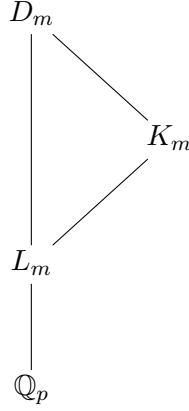
Simply take $f \in H^1(G_p, \mathbb{Q}_p(\psi))$, consider $\rho_f : G_p \rightarrow GL_2(\mathbb{Q}_p)$ given by $\rho(\tau) = \begin{pmatrix} \psi(\tau) & f(\tau) \\ 0 & 1 \end{pmatrix}$

and conjugate by an appropriate power of $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ to get the desired integral representation.

Rather than deal with ρ_ψ , we deal with its mod p^m reduction, $\rho_{\psi,m}$. Let L_m be the composite of $\mathbb{Q}_p(\mu_{p^m})$ and the \mathbb{Z}/p^{m-1} -unramified extension of \mathbb{Q}_p . Note the $*$ in $\rho_{\psi,m}$ gives rise to a cyclic extension of order p^{m-1} , not order p^m .

We treat the $\psi = \epsilon$ case separately. Here we can just take $\rho_{\epsilon,m}$ to arise from the splitting field of $x^{p^m} - (1+p)^{p^{m-1}}$.

For $\psi \neq \epsilon$, let d be the unique integer satisfying $\psi \equiv \epsilon \pmod{p^d}$ and $\psi \not\equiv \epsilon \pmod{p^{d+1}}$. Let D_m be the maximal abelian extension of L_m whose Galois group is killed by p^{m-1} . Let K_m be the composite of L_m and the field fixed by Kernel($\rho_{\psi,m}$).



The Kummer pairing $\text{Gal}(D_m/L_m) \times L_m^*/L_m^{*p^{m-1}} \rightarrow \mu_{p^{m-1}}$ is perfect and $\text{Gal}(L_m/\mathbb{Q}_p)$ -equivariant so $\text{Gal}(D_m/L_m)$ is isomorphic to the \mathbb{G}_m -dual of $L_m^*/L_m^{*p^{m-1}}$ as a $\text{Gal}(L_m/\mathbb{Q}_p)$ -module. As K_m/L_m is a cyclic extension of order p^{m-1} and K_m/\mathbb{Q}_p is Galois with $\text{Gal}(L_m/\mathbb{Q}_p)$ acting on $\text{Gal}(K_m/L_m)$ by the character ψ , we see K_m arises over L_m by adding p^{m-1} st roots of an element $\alpha \in L_m^*/(L_m^*)^{p^{m-1}}$ which, by the above \mathbb{G}_m -duality, generates a ϵ/ψ -eigenspace in this group. So we need to prove such an eigenspace exists in the *unit* part of $L_m^*/(L_m^*)^{p^{m-1}}$.

Recall $L_m^* \simeq \langle \pi_m \rangle \times U_{L_m}$ where π_m is a uniformiser of L_m and U_{L_m} is the group of units. Write $\alpha = \pi_m^{p^k a} u$ where $p \nmid a$ and $u \in U_{L_m}$ so $K_m = L_m(\alpha^{1/p^{m-1}})$. Let $\sigma \in \text{Gal}(L_m/\mathbb{Q}_p)$ and set $\sigma(\pi_m) = \pi_m w_\sigma$ and $\sigma(u) = u_\sigma$ where $w_\sigma, u_\sigma \in U_{L_m}$. We have

$$\sigma(\alpha) = \sigma(\pi_m^{p^k a} u) = \sigma(\pi_m)^{p^k a} \sigma(u) = \pi_m^{p^k a} w_\sigma^{p^k a} u_\sigma.$$

But we also have

$$\sigma(\alpha) \equiv (\alpha)^{\frac{\epsilon}{\psi}(\sigma)} \equiv (\pi_m^{p^k a} u)^{\frac{\epsilon}{\psi}(\sigma)} \pmod{(L_m^*)^{p^{m-1}}}$$

so we get

$$(\pi_m^{p^k a})^{\frac{\epsilon}{\psi}(\sigma)-1} \equiv \text{a unit} \pmod{(L_m^*)^{p^{m-1}}}.$$

This can only happen if the left side is trivial, that is the exponent of π_m is a multiple of p^{m-1} . Thus

$$p^k \left(\frac{\epsilon}{\psi}(\sigma) - 1 \right) \equiv 0 \pmod{p^{m-1}}.$$

Since σ is arbitrary, the definition of d implies $k + d \geq m - 1$ so $k \geq m - d - 1$. Thus when we take the p^{m-d-1} st root of α and adjoin this to L_m , we are taking the root of a unit. So $\rho_{\psi, m-1-d}$ arises as desired, by taking the p -power root of a unit. Now simply let $m \rightarrow \infty$. \square

It is possible to build mod p^m representations that are extensions of 1 by ψ that do arise by taking p^{m-1} st roots of nonunits of L_m . The proof of Lemma 4 shows such extensions, however, do not lift to characteristic zero when $\psi \neq \epsilon$.

Lemma 5. *Let the hypotheses be as in Lemma 4 and let $\psi = \epsilon$. Then $\text{Kernel}(\rho_{\epsilon, m})$ fixes the splitting field of $x^{p^m} - a$ for some $a \in \mathbb{Z}_p$. This Galois representation corresponds to finite flat group scheme over \mathbb{Z}_p if and only if $a \in \mathbb{Z}_p^*$.*

Proof. We only sketch the proof.

It is (again) well-known that $H^1(G_p, \mathbb{Z}_p/(p^m)(\epsilon)) \simeq (\mathbb{Z}_p/(p^m))^2$. The representations associated to the splitting fields of $x^{p^m} - p$ and $x^{p^m} - (1 + p)$ correspond to cohomology classes that form a basis for this module.

Since $* \equiv 0 \pmod{p}$ and $* \not\equiv 0 \pmod{p^2}$ we have that a is a p th power in \mathbb{Z}_p but not a p^2 th power.

If a is a unit it is clear that the G_p -module corresponding to $\rho_{\epsilon, m}$ comes from a finite flat group scheme over \mathbb{Z}_p .

Using Fontaine-Lafaille theory, [5], one can count how many extensions of $\mathbb{Z}/(p^m)(\epsilon)$ by $\mathbb{Z}/(p^m)$ there are in the category of finite flat group schemes over \mathbb{Z}_p up to isomorphism. One finds $m - 1$ of them where the $*$ is as above. These correspond to $a = (1 + p)^p, (1 + p)^{p^2}, \dots, (1 + p)^{p^{m-1}}$, all of which are units. \square

Proposition 6. *Let $\bar{\eta} : G_p \rightarrow \mathbb{F}_q^*$ be a nontrivial unramified character. Set $h^0 = \dim H^0(G_p, \text{Ad}\bar{\rho})$. Up to twist we have the the following possibilities for $\bar{\rho}|_{G_p}$ and its **local** deformation ring:*

- 1) $\bar{\rho}|_{G_p} = \begin{pmatrix} \bar{\eta}\bar{\epsilon} & 0 \\ 0 & 1 \end{pmatrix}$. Here $\dim \tilde{\mathcal{L}}_p = 4$, $\dim H^0(G_p, \text{Ad}\bar{\rho}) = 2$ and the ordinary deformation ring is smooth in 4 variables.
- 2) $\bar{\rho}|_{G_p} = \begin{pmatrix} \bar{\eta}\bar{\epsilon} & * \\ 0 & 1 \end{pmatrix}$. Here $\dim \tilde{\mathcal{L}}_p = 3$, $\dim H^0(G_p, \text{Ad}\bar{\rho}) = 1$ and the ordinary deformation ring is smooth in 3 variables.
- 3) $\bar{\rho}|_{G_p} = \begin{pmatrix} \bar{\epsilon} & * \\ 0 & 1 \end{pmatrix}$ is flat. Here $\dim \tilde{\mathcal{L}}_p = 3$, $\dim H^0(G_p, \text{Ad}\bar{\rho}) = 1$ and the ordinary deformation ring is smooth in 3 variables.
- 4) $\bar{\rho}|_{G_p} = \begin{pmatrix} \bar{\epsilon} & * \\ 0 & 1 \end{pmatrix}$ is not flat. Here $\dim \tilde{\mathcal{L}}_p = 3$, $\dim H^0(G_p, \text{Ad}\bar{\rho}) = 1$ and the ordinary deformation ring is smooth in 3 variables.
- 5) $\bar{\rho}|_{G_p} = \begin{pmatrix} \bar{\epsilon} & 0 \\ 0 & 1 \end{pmatrix}$. Here $\dim \tilde{\mathcal{L}}_p = 5$ and the ordinary deformation ring is not smooth, but it has a smooth quotient in four variables whose characteristic zero points include all points of weight $k > 2$ and all flat points of weight $k = 2$. So we redefine $\tilde{\mathcal{L}}_p$ to be the 4 dimensional subspace induced by this quotient and note $\dim H^0(G_p, \text{Ad}\bar{\rho}) = 2$.

Proof. Let $U \subset \text{Ad}\bar{\rho}$ be the upper triangular matrices. In each case we will compare the unrestricted (local) upper triangular deformation theory of $\bar{\rho}|_{G_p}$ to its (local) ordinary deformation theory.

1) $\tilde{\mathcal{L}}_p$ includes the two unramified twists, the one ramified twist of $\bar{\eta}\bar{\epsilon}$ and the nontrivial extension of 1 by $\bar{\eta}\bar{\epsilon}$ so $\dim \tilde{\mathcal{L}}_p = 4$. One easily sees that $\dim H^2(G_p, U) = 0$ and $\dim H^1(G_p, U) = 5$ so the upper triangular deformation ring is smooth in 5 variables. Its ordinary quotient is formed by forcing the lower right entry to be unramified. This involves the one relation that comes from setting the ramified part of the lower right entry, when evaluated at a topological generator of the Galois group over \mathbb{Q}_p of the cyclotomic extension, to be trivial. Since this relation necessarily cuts the tangent space down by one variable, we can take it to be a variable of the 5 dimensional upper triangular ring, so the ordinary ring is smooth in 4 variables.

2) One computes $\dim H^2(G_p, U) = 0$ and $\dim H^1(G_p, U) = 4$ so the upper triangular deformation ring is smooth in 4 variables. Let U^1 be the matrices of the form $\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}$. One computes $\dim H^1(G_p, U^1) = 2$ and from Table 3 of [13] we have $\dim \mathcal{L}_p = 1$. As $\mathcal{L}_p \subset H^1(G_p, U^1)$ any element of $H^1(G_p, U^1)$ not in \mathcal{L}_p is ramified on both diagonals. A linear combination of this class and the ramified twist will be trivial on the lower right entry and thus ordinary. Of course the unramified twist is in $\tilde{\mathcal{L}}_p$ so $\dim \tilde{\mathcal{L}}_p = 3$. That the ordinary ring is smooth in 3 variables follows from the argument in the proof of 1).

3) That $\dim \tilde{\mathcal{L}}_p = 3$ follows from Proposition 13 of [15]. One easily sees that $\dim H^2(G_p, U) = 0$ and $\dim H^1(G_p, U) = 4$ so the upper triangular deformation ring is smooth in 4 variables. As before its ordinary quotient involves one relation that forces the lower right entry to be unramified which again implies the ordinary ring is smooth in 3 variables.

4) Then the short exact sequence $0 \rightarrow U^1 \rightarrow U \rightarrow U/U^1 \rightarrow 0$ and routine Galois cohomology computations give that $H^1(G_p, U^1) \rightarrow H^1(G_p, U)$ is an injection of a 2-dimensional space into a 4-dimensional space. The cokernel is spanned by the images of the ramified and unramified twists. There are two independent extensions of 1 by $\bar{\epsilon}$ so at least one dimension of $H^1(G_p, U^1)$ is ordinary.

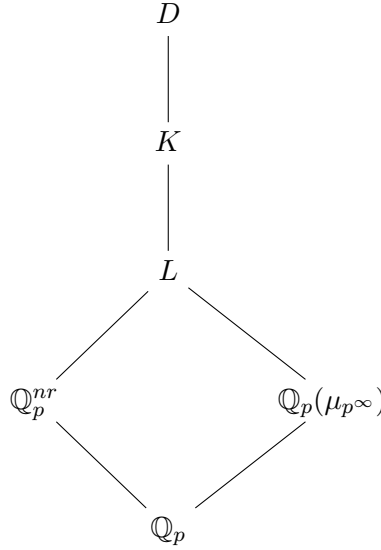
If all of $H^1(G_p, U^1) \subset H^1(G_p, U)$ is ordinary then, taking into account the unramified twist, $\dim \tilde{\mathcal{L}}_p \geq 3$. The only way $\dim \tilde{\mathcal{L}}_p = 4 = \dim H^1(G_p, U)$ is if the ramified twist belongs to $\tilde{\mathcal{L}}_p$, which we know does not happen. Thus $\dim \tilde{\mathcal{L}}_p = 3$ in this case.

If only one dimension of $H^1(G_p, U^1)$ is ordinary (this is what actually happens, but proving it is messier than the weaker argument used here) then the same proof as in 2) implies $\dim \tilde{\mathcal{L}}_p = 3$.

Since $\dim \tilde{\mathcal{L}}_p = 3$ in all cases and the upper triangular ring is smooth in 4 variables, the ordinary ring is smooth in 3 variables.

5) This case is a bit more involved as the ordinary ring is not smooth. First note that $\tilde{\mathcal{L}}_p$ contains the two unramified twists, the ramified twist of $\bar{\epsilon}$ and the *two* extensions of 1 by $\bar{\epsilon}$ and so $\dim \tilde{\mathcal{L}}_p = 5$. We will replace it by a 4-dimensional subspace.

Let D be the maximal pro- p abelian extension of L , the the composite of \mathbb{Z}_p -unramified extension of \mathbb{Q}_p and $\mathbb{Q}_p(\mu_{p^\infty})$. Then in this case, any ordinary deformation of $\bar{\rho}$ has meta-abelian image and factors through $\text{Gal}(D/\mathbb{Q}_p)$. By Kummer theory D is generated by p -power roots of elements of L . Let $K \subset D$ be the subfield generated by p -power roots of *units* of L .



Let $\psi_1, \psi_2 : G_p \rightarrow \mathbb{Z}_p^*$ be unramified characters each congruent to 1 mod p .

We will consider a series of ring homomorphisms

$$R^{\text{ord}} \twoheadrightarrow R^{\text{ord}, \text{unit}} \twoheadrightarrow R_k^{\text{ord}, \text{unit}} \twoheadrightarrow R_k^{\text{ord}, \text{unit}, \psi_2} \twoheadrightarrow R_k^{\text{ord}, \text{unit}, \psi_1, \psi_2}$$

where the superscript ‘unit’ indicates the quotient of the ordinary ring whose deformation factors through $\text{Gal}(K/\mathbb{Q}_p)$ and the presence of the unramified character ψ_i as a superscript indicates that we are fixing ψ_i in the ii spot on the diagonal. The subscript k indicates the weight. So $R_k^{\text{ord}, \text{unit}, \psi_1, \psi_2}$ is the deformation ring parametrizing deformations of $\bar{\rho}|_{G_p}$ that factor through $\text{Gal}(K/\mathbb{Q}_p)$ and are of the form $\begin{pmatrix} \epsilon^{k-1}\bar{\epsilon}^{2-k}\psi_1 & * \\ 0 & \psi_2 \end{pmatrix}$. For instance, the ring $R_k^{\text{ord}, \text{unit}}$ puts no restrictions on the unramified diagonal characters.

Consider $R_k^{\text{ord}, \text{unit}, \psi_1, \psi_2}$. The tangent space for this ring is 1-dimensional as follows. No twists by characters on the diagonal are allowed and the très ramifiée extension of 1 by $\bar{\epsilon}$ is not allowed either. Only the peu ramifiée extension of 1 by $\bar{\epsilon}$ is allowed. Thus the corresponding deformation ring is $\mathbb{Z}_p[[U]]/I_1$. If I_1 contains a nonzero element $g(U)$, then by the Weierstrass preparation theorem we can assume $g(U) = p^r h(U)$ where $h(U)$ is a distinguished polynomial, or $h(U) \equiv 1$ or 0. But Lemma 4 gives the existence of nonsplit characteristic zero deformations. Conjugating these by

$\begin{pmatrix} p^m & 0 \\ 0 & 1 \end{pmatrix}$ gives *different deformations* of $\bar{\rho}$ for each m (though of course these representations are all isomorphic), so our ring has infinitely many characteristic zero points and $h(U)$ would have infinitely many roots, a contradiction. Thus $h(U) \equiv 0$, I_1 is trivial and $R_k^{ord,unit,\psi_1,\psi_2} \simeq \mathbb{Z}_p[[U]]$.

The ring $R_k^{ord,unit,\psi_2}$ has two dimensional tangent space (the unramified twist of $\bar{\epsilon}$ is now allowed) and so $R_k^{ord,unit,\psi_2} \simeq \mathbb{Z}_p[[U_1, U_2]]/I_2$. But for each choice of ψ_1 , we see this ring has a different quotient isomorphic to $\mathbb{Z}_p[[U]]$. If $I_2 \neq (0)$ Krull's principal ideal theorem (see Corollary 11.18 of [1]) implies $R_k^{ord,unit,\psi_2}$ has Krull dimension at most 2. Then a Noetherian ring of Krull dimension at most 2 has infinitely many components of Krull dimension 2, a contradiction. Thus $I_2 = 0$. Similarly $R_k^{ord,unit}$ has three dimensional tangent space and so $R_k^{ord,unit} \simeq \mathbb{Z}_p[[U_1, U_2, U_3]]/I_3$. But for each choice of ψ_2 , we see this ring has a different quotient isomorphic to $\mathbb{Z}_p[[U_1, U_2]]$. If $I_3 \neq (0)$ the same Krull dimension argument as above gives a contradiction. Thus $I_3 = 0$. Finally, $R^{ord,unit}$ has four dimensional tangent space as only the très ramifiée extension of 1 by $\bar{\epsilon}$ is not allowed. We have, for each $k \geq 2$,

$$R^{ord,unit} \twoheadrightarrow R_k^{ord,unit} \simeq \mathbb{Z}_p[[U_1, U_2, U_3]]$$

so, arguing as before, $R^{ord,unit} \simeq \mathbb{Z}_p[[U_1, U_2, U_3, U_4]]$.

Using Lemma 4 we see it remains to show that weight 2 flat deformations of $\bar{\rho}$ factor through $R^{ord,unit}$ in the $\psi_1 = \psi_2 = \psi$ case. This follows immediately from Lemma 5. \square

Proposition 7. *For $v \neq p$, $\dim \tilde{\mathcal{L}}_v = \dim H^0(G_v, Ad\bar{\rho})$ and $\dim \tilde{\mathcal{L}}_p = \dim H^0(G_p, Ad\bar{\rho}) + 2$*

Proof. For $v \neq p$ it is known that $\dim \mathcal{L}_v = \dim H^0(G_v, Ad^0\bar{\rho})$. As we switch from $Ad^0\bar{\rho}$ to $Ad\bar{\rho}$

$$\dim H^0(G_v, Ad\bar{\rho}) - \dim H^0(G_v, Ad^0\bar{\rho}) = 1 = \dim \tilde{\mathcal{L}}_v - \dim \mathcal{L}_v.$$

The $v = p$ result follows from Fact 6. \square

Propositions 2 and 7 give, taking into account $v = \infty$,

Corollary 8. $\dim H_{\tilde{\mathcal{L}}}^1(G_S, Ad\bar{\rho}) - \dim H_{\tilde{\mathcal{L}}^\perp}^1(G_S, Ad\bar{\rho}^*) = 1$.

3.2. Local deformation rings at nice primes. Finally, we recall the notion of *nice* primes for a representation. The definition given below is a blend of that given in [14] (see §2 and Propostion 2.2) and that of [18] that is suited for our purposes. The latter reference handles the case $p = 3$.

Definition 9. *Let $\bar{\rho} : G_{\mathbb{Q}} \rightarrow GL_2(\mathbb{F}_{p^f})$ odd, ordinary, full and weight 2 trivial nebentype be given. Let R be a finite cardinality complete local Noetherian ring with residue field \mathbb{F}_{p^f} and let ρ_R be a continuous homomorphism $G_{\mathbb{Q}} \rightarrow GL_2(R)$ lifting $\bar{\rho}$. A prime $q \not\equiv 1 \pmod{p}$ is called **nice** if $\bar{\rho}$ is unramified at q and if the eigenvalues of $\bar{\rho}(Fr_q)$ are q and 1. The prime q is called **ρ_R -nice** if ρ_R is unramified at q , $\rho_R(Fr_q)$ has eigenvalues q and 1 and order prime to p .*

The local at q deformation ring has a smooth quotient whose points \mathcal{C}_q consist of Steinberg deformations. There is an induced subspace $\mathcal{L}_q \subset H^1(G_q, Ad^0\bar{\rho})$. When $q \not\equiv \pm 1 \pmod{p}$, there is a single family of Steinberg deformations. When $q \equiv -1 \pmod{p}$, a necessity in the $p = 3$ case, there are two families of Steinberg deformations. Taylor simply chooses a family and defines \mathcal{L}_q and \mathcal{C}_q accordingly. See [18] for all of this.

Proposition 10. *Let ρ_R be odd, ordinary, weight 2 with full reduction $\bar{\rho}$. Then ρ_R -nice primes exist and for any nice prime q , $\dim \mathcal{L}_q = \dim H^0(G_q, Ad^0\bar{\rho}) = 1$ and a smooth quotient of the deformation ring exists with points \mathcal{C}_q . Proposition 7 applies for nice primes so $\dim \tilde{\mathcal{L}}_q = 2 = \dim H^0(G_q, Ad\bar{\rho})$ and $\tilde{\mathcal{C}}_q$ consists of all unramified twists of points of \mathcal{C}_q .*

Proof. We are given that $\bar{\rho}$ is full and $\det \bar{\rho} = \bar{\epsilon}$ so for $a \in \mathbb{F}_p$, $a \neq 1$, choose $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in \text{image}(\bar{\rho})$. Any prime q with Frobenius in the conjugacy class of this matrix will be nice. After lifting this matrix to an element of $\text{image}(\rho_R)$ and raising to a large power of p (say p^r), the new matrix will have tame order and eigenvalues $\{a^{p^r}, 1\}$ which are distinct. Any prime with Frobenius in the conjugacy class of this element will be ρ_R -nice.

The cohomological results are standard and we do not give their proofs. \square

4. ORDINARY SMOOTH DEFORMATION RINGS AND ρ_n

For any finite set of primes $T \supset S$ with $T \setminus S$ consisting of only nice primes, we have an ordinary arbitrary weight deformation ring denoted $R^{\text{ord}, T-\text{new}}$ and its weight 2 quotient $R_2^{\text{ord}, T-\text{new}}$. When restricted to G_v the points of these rings lie, respectively, in $\tilde{\mathcal{C}}_v$ and \mathcal{C}_v . We remark that in previous papers we used the notation $T \setminus S_0$ -new to indicate that all nice primes were in the level of the modular form. Since it is less cumbersome, we use T -new here instead. Results toward Theorem 11 are proved in [3].

Theorem 11. *Suppose $\rho_n : G_{\mathbb{Q}} \rightarrow GL_2(\mathcal{O}/(\pi^n))$ is odd, ordinary, weight 2, modular, has full image and determinant ϵ , and is balanced. Suppose $\rho_n|_{G_v} \in \mathcal{C}_v$ for all $v \in S$. Then there exists a finite set of primes $T \supseteq S$ such that the universal ordinary ‘new at T ’ ring $R^{\text{ord}, T-\text{new}} \simeq W(\mathbb{F}_{p^f})[[U]]$ and there are surjections*

$$R^{\text{ord}, T-\text{new}} \twoheadrightarrow R_2^{\text{ord}, T-\text{new}} \twoheadrightarrow \mathcal{O}/(\pi^n)$$

from this ring to its weight 2 quotient and then to $\mathcal{O}/(\pi^n)$ inducing ρ_n .

It is **not** a consequence of Theorem 11 that ρ_n lifts to a T -new weight 2 characteristic zero representation. For instance, if $n = 3$ and $\mathcal{O} = \mathbb{Z}_p[\sqrt{p}]$ it is possible that

$$R_2^{\text{ord}, T-\text{new}} \simeq \mathbb{Z}_p[[U]] / ((U - p)(U - 2p)(U - 3p))$$

and ρ_3 arises from $U \mapsto \sqrt{p}$. The smoothness of $R^{\text{ord}, T-\text{new}}$ immediately implies the existence of characteristic zero lifts, but these lifts may not have classical weight, let alone weight 2. Theorem 20 addresses this.

4.1. Group theoretic lemmas. We need the following lemma of Boston (see [2]) and Lemma 13 for Lemma 14.

Lemma 12. *(Boston) Let $p \geq 3$. Let R be a complete local Noetherian ring with residue characteristic p . Let $\rho : G \rightarrow GL_2(R)$ be a representation and assume the image of the projection*

$$\rho_2 : G \rightarrow GL_2(R) \rightarrow GL_2(R/m_R^2)$$

is full, that is it contains $SL_2(R/m_R^2)$. Then the image of ρ contains $SL_2(R)$.

Lemma 13 below is well-known, but as we could find no precise reference we include a proof.

Lemma 13. *Let $G \subset GL_2(\mathbb{F}_{p^f})$ be a full subgroup. If $\mathbb{F}_{p^f} \neq \mathbb{F}_5$ then $H^1(G, \text{Ad}^0 \bar{\rho}) = 0$. If $\mathbb{F}_{p^f} = \mathbb{F}_5$, suppose there exists $\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \in G$ with $\frac{r}{s} \neq \frac{s}{r}$. Then $H^1(G, \text{Ad}^0 \bar{\rho}) = 0$.*

Proof. That $H^1(SL_2(\mathbb{F}_3), \text{Ad}^0 \bar{\rho}) = 0$ is well-known. See for instance Lemma 1 of [11]. Since $3 \nmid [G : SL_2(\mathbb{F}_3)]$ the restriction map $H^1(G, \text{Ad}^0 \bar{\rho}) \rightarrow H^1(SL_2(\mathbb{F}_3), \text{Ad}^0 \bar{\rho})$ is injective and the result follows for \mathbb{F}_3 .

Let B be the upper triangular Borel subgroup of G and let N be its unipotent subgroup. Fullness implies $p \nmid [G : B]$ so again the restriction map $H^1(G, \text{Ad}^0 \bar{\rho}) \rightarrow H^1(B, \text{Ad}^0 \bar{\rho})$ is injective. We will prove $H^1(B, \text{Ad}^0 \bar{\rho}) = 0$.

Let $U^0 \subset U^1 \subset \text{Ad}^0 \bar{\rho}$ where U^0 is the space of upper triangular nilpotent matrices and U^1 is the space of upper triangular trace zero matrices. From the short exact sequence

$$0 \rightarrow U^1 \rightarrow \text{Ad}^0 \bar{\rho} \rightarrow \text{Ad}^0 \bar{\rho}/U^1 \rightarrow 0$$

we easily get

$$\dots \rightarrow 0 \rightarrow H^1(B, U^1) \rightarrow H^1(B, \text{Ad}^0 \bar{\rho}) \rightarrow H^1(B, \text{Ad}^0 \bar{\rho}/U^1) \rightarrow \dots$$

so it suffices to prove the (nonzero) outside terms are trivial.

Associated to $H^1(B, \text{Ad}^0 \bar{\rho}/U^1)$ we the exact inflation-restriction sequence

$$0 \rightarrow H^1(B/N, (\text{Ad}^0 \bar{\rho}/U^1)^N) \rightarrow H^1(B, \text{Ad}^0 \bar{\rho}/U^1) \rightarrow H^1(N, \text{Ad}^0 \bar{\rho}/U^1)^{B/N} \rightarrow H^2(B/N, (\text{Ad}^0 \bar{\rho}/U^1)^N).$$

As $p \nmid |B/N|$ the outside terms are trivial. One easily sees N acts trivially on $\text{Ad}^0 \bar{\rho}/U^1$ so $H^1(N, \text{Ad}^0 \bar{\rho}/U^1)^{B/N} = \text{Hom}_{B/N}(N, \text{Ad}^0 \bar{\rho}/U^1)$. Observe $\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \in B/N$ acts on N by $\frac{r}{s}$ and on the 1-dimensional space $\text{Ad}^0 \bar{\rho}/U^1$ by $\frac{s}{r}$. As long as there is an element as above where $\frac{r}{s} \neq \frac{s}{r}$, we have $H^1(N, \text{Ad}^0 \bar{\rho}/U^1)^{B/N} = 0$ so $H^1(B, \text{Ad}^0 \bar{\rho}/U^1) = 0$. Choose $r = a$ and $s = a^{-1}$ so we are requiring $a^4 \neq 1$. Such a exist in \mathbb{F}_{p^f} when $p^f \neq 3, 5$. The extra hypothesis for $p^f = 5$ gives that $H^1(B, \text{Ad}^0 \bar{\rho}/U^1) = 0$ in this last case.

For the $H^1(B, U^1)$ term, consider the short exact sequence

$$0 \rightarrow U^0 \rightarrow U^1 \rightarrow U^1/U^0 \rightarrow 0$$

and take its B -cohomology to get

$$\dots 0 \rightarrow \mathbb{F}_{p^f} \rightarrow H^1(B, U^0) \rightarrow H^1(B, U^1) \rightarrow H^1(B, U^1/U^0).$$

An easy analysis of the exact inflation-restriction sequences

$$0 \rightarrow H^1(B/N, (U^0)^N) \rightarrow H^1(B, U^0) \rightarrow H^1(N, U^0)^{B/N} \rightarrow H^2(B/N, (U^0)^N)$$

and

$$0 \rightarrow H^1(B/N, (U^1/U^0)^N) \rightarrow H^1(B, U^1/U^0) \rightarrow H^1(N, U^1/U^0)^{B/N} \rightarrow H^2(B/N, (U^1/U^0)^N)$$

gives that $\dim H^1(B, U^0) = 1$ and $\dim H^1(B, U^1/U^0) = 0$. Thus $H^1(B, U^1) = 0$ and the proof is complete. \square

Lemma 14. *Let $p \geq 3$. Let $G \subset GL_2(\mathcal{O}/(\pi^r))$ be a subgroup. Suppose the hypotheses of Lemma 13 are satisfied for the image of $G \rightarrow GL_2(\mathbb{F}_{p^f})$ and that the image of the projection $p_2 : G \rightarrow GL_2(\mathcal{O}/(\pi^2))$ is full. Then $\dim H^1(G, \text{Ad}^0 \bar{\rho}) = 1$.*

Proof. Since the image of p_2 is full the hypothesis of Lemma 12 is satisfied so $G \supset SL_2(\mathcal{O}/(\pi^r))$.

Let Γ be the kernel of the projection $G \rightarrow GL_2(\mathbb{F}_{p^f})$. We have the exact inflation-restriction sequence

$$0 \rightarrow H^1(G/\Gamma, \text{Ad}^0 \bar{\rho}^\Gamma) \rightarrow H^1(G, \text{Ad}^0 \bar{\rho}) \rightarrow H^1(\Gamma, \text{Ad}^0 \bar{\rho})^{G/\Gamma}.$$

As G/Γ is the image of the projection $p_1 : G \rightarrow GL_2(\mathbb{F}_{p^f})$, Lemma 13 implies the first term is trivial.

Also, as Γ acts trivially on $\text{Ad}^0 \bar{\rho}$,

$$H^1(\Gamma, \text{Ad}^0 \bar{\rho})^{G/\Gamma} = \text{Hom}_{G/\Gamma}(\Gamma, \text{Ad}^0 \bar{\rho}).$$

For any $\gamma \in \text{Hom}_{G/\Gamma}(\Gamma, \text{Ad}^0 \bar{\rho})$, $\text{Kernel}(\gamma) \supset \Gamma'$, the commutator subgroup of Γ .

Set $a = 1 + \pi$ and $r = \frac{\pi x}{2 + \pi}$ and use fullness to see

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}^{-1} \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & \pi^2 x \\ 0 & 1 \end{pmatrix} \in \Gamma' \subset \text{Kernel}(\gamma).$$

Similarly,

$$\begin{pmatrix} 1 & 0 \\ \pi^2 y & 1 \end{pmatrix} \in \Gamma' \subset \text{Kernel}(\gamma).$$

As $\text{Kernel}(\gamma)$ is stable under the action of SD ,

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \pi^2 z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 - \pi^2 z & \pi^2 z \\ -\pi^2 z & 1 + \pi^2 z \end{pmatrix} \subset \text{Kernel}(\gamma).$$

Multiplying on the left and right by suitable matrices

$$\begin{pmatrix} 1 & 0 \\ \pi^2 y & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & \pi^2 x \\ 0 & 1 \end{pmatrix}$$

we have

$$\begin{pmatrix} 1 + \pi^2 z & 0 \\ 0 & (1 + \pi^2 z)^{-1} \end{pmatrix} \in \text{Kernel}(\gamma).$$

As every element of

$$\Gamma_2 := \{A \in SL_2(\mathcal{O}/(\pi^r)) \mid A \equiv I \pmod{(\pi^2)}\}$$

can be written as a product

$$A = \begin{pmatrix} 1 & 0 \\ \pi^2 y & 1 \end{pmatrix} \begin{pmatrix} 1 + \pi^2 z & 0 \\ 0 & (1 + \pi^2 z)^{-1} \end{pmatrix} \begin{pmatrix} 1 & \pi^2 x \\ 0 & 1 \end{pmatrix}$$

we have $\text{Kernel}(\gamma) \supset \Gamma_2$. Since $\Gamma/\Gamma_2 \simeq \text{Ad}^0 \bar{\rho}$,

$$\dim \text{Hom}_{G/\Gamma}(\Gamma, \text{Ad}^0 \bar{\rho}) \leq \dim \text{Hom}_{G/\Gamma}(\Gamma/\Gamma_2, \text{Ad}^0 \bar{\rho}) = \dim \text{Hom}_{G/\Gamma}(\text{Ad}^0 \bar{\rho}, \text{Ad}^0 \bar{\rho}) = 1$$

so $\dim H^1(G, \text{Ad}^0 \bar{\rho}) \leq 1$. As \mathcal{O}/\mathbb{Z}_p is ramified, $GL_2(\mathcal{O}/(\pi^2)) \simeq GL_2(\mathbb{F}_{p^f}[U]/(U^2))$ is nontrivial as we are given full image so $\dim H^1(G, \text{Ad}^0 \bar{\rho}) = 1$. \square

4.2. Selmer goup. All sets of primes Z below will be finite, contain S and $Z \setminus S$ will consist of nice primes.

Proposition 15. *Let $h \in H_{\tilde{\mathcal{L}}}^1(G_Z, \text{Ad} \bar{\rho})$ and $\phi \in H_{\tilde{\mathcal{L}}^\perp}^1(G_S, \text{Ad} \bar{\rho}^*)$. If $h \in H^1(G_Z, \mathbb{F}_{p^f}) \subset H^1(G_Z, \text{Ad} \bar{\rho})$ then $h = 0$. If $\phi \in H^1(G_Z, \mathbb{F}_{p^f}(1)) \subset H^1(G_Z, \text{Ad} \bar{\rho}^*)$ then $\phi = 0$.*

Proof. If $h \in H^1(G_Z, \mathbb{F}_{p^f})$, it corresponds, when viewed as a lift to the dual numbers, to a twist by a character which gives rise to a $\mathbb{Z}/(p)$ -extension of \mathbb{Q} . Definition 3 implies that $\tilde{\mathcal{L}}_v \cap H^1(G_v, \mathbb{F}_{p^f})$ is spanned, for all v , by the \mathbb{F}_{p^f} -valued unramified twists so the corresponding global extension is unramified everywhere so $h = 0$.

Set $M = \mathbb{F}_{p^f}$ and for all v set

$$\mathcal{M}_v = \tilde{\mathcal{L}}_v \cap H^1(G_v, \mathbb{F}_{p^f}) = H_{nr}^1(G_v, \mathbb{F}_{p^f}).$$

We just showed $H_{\mathcal{M}}^1(G_Z, M) = 0$. As $\dim \mathcal{M}_v = \dim H^0(G_v, M) = 1$ for $v \neq \infty$, Proposition 2 gives $\dim H_{\mathcal{M}^\perp}^1(G_Z, M^*) = 0$ as well.

Any $\phi \in H^1(G_Z, \mathbb{F}_{p^f}(1)) \cap H_{\tilde{\mathcal{L}}^\perp}^1(G_Z, \text{Ad} \bar{\rho}^*)$ cuts out an extension $L/\mathbb{Q}(\mu_p)$ that is Galois over \mathbb{Q} and $\text{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})$ acts on $\text{Gal}(L/\mathbb{Q}(\mu_p))$ by $\bar{\epsilon}$. At $v \neq p$ unramified cohomologies are exact annihilators under the local duality pairing so $\tilde{\mathcal{L}}_v^\perp \cap H^1(G_v, \mathbb{F}_{p^f}(1)) = H_{nr}^1(G_v, \mathbb{F}_{p^f}(1))$. (This last group is trivial unless $v \equiv 1 \pmod{p}$). So for $v \neq p$, $L/\mathbb{Q}(\mu_p)$ is unramified at v .

For $v = p$ choose a subspace $V \subset H^1(G_p, \text{Ad} \bar{\rho})$ such that

$$\tilde{\mathcal{L}}_p = (\mathcal{L}_p \oplus \mathcal{M}_p) + V, \quad V \cap (\mathcal{L}_p \oplus \mathcal{M}_p) = 0$$

so

$$\tilde{\mathcal{L}}_p^\perp = (\mathcal{L}_p \oplus \mathcal{M}_p)^\perp \cap V^\perp = (\mathcal{L}_p^\perp \oplus \mathcal{M}_p^\perp) \cap V^\perp$$

and thus

$$\tilde{\mathcal{L}}_p^\perp \cap H^1(G_p, \mathbb{F}_{p^f}(1)) \subset \mathcal{M}_p^\perp.$$

Thus $\phi|_{G_v} \in \mathcal{M}_v^\perp$ for all v , that is $\phi \in H_{\mathcal{M}^\perp}^1(G_Z, M)$ which we already proved is trivial so $\phi = 0$. \square

Note that for $h \in H^1(G_Z, Ad^0 \bar{\rho})$ and $q \notin Z$ nice, $h|_{G_q} \neq 0$ is equivalent to $h|_{G_q} \notin \mathcal{L}_q$. Similarly, for $h \in H^1(G_Z, Ad\bar{\rho})$ write $h = h_{Ad^0 \bar{\rho}} + h_{sc}$ with $h_{Ad^0 \bar{\rho}} \in H^1(G_Z, Ad^0 \bar{\rho})$ and $h_{sc} \in H^1(G_Z, \mathbb{F}_{p^f})$. For $q \notin Z$ nice, $h|_{G_q} \notin \tilde{\mathcal{L}}_q$ is equivalent to $h_{Ad^0 \bar{\rho}}|_{G_q} \notin \mathcal{L}_q$ which we just saw is equivalent to $h_{Ad^0 \bar{\rho}}|_{G_q} \neq 0$.

Proposition 16. *Let $h \in H_{\mathcal{L}}^1(G_Z, Ad^0 \bar{\rho})$, $\phi \in H_{\mathcal{L}^\perp}^1(G_Z, Ad^0 \bar{\rho}^*)$ and let q be nice.*

1) *The injective inflation map*

$$H^1(G_Z, Ad^0 \bar{\rho}) \rightarrow H^1(G_{Z \cup \{q\}}, Ad^0 \bar{\rho})$$

has codimension 0 or 1. If $\text{III}^1(Ad^0 \bar{\rho}^)|_{G_q} = 0$ then the codimension is 1.*

2) *If $\phi|_{G_q} \neq 0$ then the maps*

$$H^1(G_{Z \cup \{q\}}, Ad^0 \bar{\rho}) \rightarrow \oplus_{v \in Z} \frac{H^1(G_v, Ad^0 \bar{\rho})}{\mathcal{L}_v} \text{ and } H^1(G_Z, Ad^0 \bar{\rho}) \rightarrow \oplus_{v \in Z} \frac{H^1(G_v, Ad^0 \bar{\rho})}{\mathcal{L}_v}$$

have the same kernel.

3) *If $h, \phi|_{G_q} \neq 0$ then*

$$\dim H_{\mathcal{L}}^1(G_{Z \cup \{q\}}, Ad^0 \bar{\rho}) = H_{\mathcal{L}}^1(G_Z, Ad^0 \bar{\rho}) - 1, \dim H_{\mathcal{L}^\perp}^1(G_{Z \cup \{q\}}, Ad^0 \bar{\rho}^*) = \dim H_{\mathcal{L}^\perp}^1(G_Z, Ad^0 \bar{\rho}^*) - 1.$$

4) *If $H_{\mathcal{L}}^1(G_Z, Ad^0 \bar{\rho})|_{G_q} = 0$, $\phi|_{G_q} \neq 0$ then*

$$H_{\mathcal{L}}^1(G_{Z \cup \{q\}}, Ad^0 \bar{\rho}) = H_{\mathcal{L}}^1(G_Z, Ad^0 \bar{\rho}), \dim H_{\mathcal{L}^\perp}^1(G_{Z \cup \{q\}}, Ad^0 \bar{\rho}^*) = \dim H_{\mathcal{L}^\perp}^1(G_Z, Ad^0 \bar{\rho}^*).$$

5) *If $H^1(G_Z, Ad^0 \bar{\rho}^*)|_{G_q} = 0$ then*

$$H^1(G_{Z \cup \{q\}}, Ad^0 \bar{\rho}) \rightarrow \oplus_{v \in Z} \frac{H^1(G_v, Ad^0 \bar{\rho})}{\mathcal{L}_v}$$

and

$$H^1(G_Z, Ad^0 \bar{\rho}) \rightarrow \oplus_{v \in Z} \frac{H^1(G_v, Ad^0 \bar{\rho})}{\mathcal{L}_v}$$

have the same image.

Proof. As the proofs of all parts are similar, we only prove part 2). We use the normal local Selmer condition for $v \in Z$, but just for this proof we set $\mathcal{L}_q = H^1(G_q, Ad^0 \bar{\rho})$ so $\mathcal{L}_q^\perp = 0$. We apply Proposition 2 with the sets Z and $Z \cup \{q\}$. Then

$$\dim H_{\mathcal{L}}^1(G_{Z \cup \{q\}}, Ad^0 \bar{\rho}) - \dim H_{\mathcal{L}^\perp}^1(G_{Z \cup \{q\}}, Ad^0 \bar{\rho}^*) = \dim H_{\mathcal{L}}^1(G_Z, Ad^0 \bar{\rho}) - \dim H_{\mathcal{L}^\perp}^1(G_Z, Ad^0 \bar{\rho}^*) + 1.$$

As $\mathcal{L}_q^\perp = 0$ we have $H_{\mathcal{L}^\perp}^1(G_{Z \cup \{q\}}, Ad^0 \bar{\rho}^*) \subset H_{\mathcal{L}^\perp}^1(G_Z, Ad^0 \bar{\rho}^*)$ and since $\phi|_{G_q} \neq 0$ this containment is proper. As dual Selmer goes down by 1 in dimension as we switch from Z to $Z \cup \{q\}$ the above equation implies the dimension of Selmer does not change as we switch from Z to $Z \cup \{q\}$. Since $\mathcal{L}_q = H^1(G_q, Ad^0 \bar{\rho})$ we have $H_{\mathcal{L}}^1(G_Z, Ad^0 \bar{\rho}) \subset H_{\mathcal{L}}^1(G_{Z \cup \{q\}}, Ad^0 \bar{\rho})$ and the result follows. \square

Proposition 17. *Let $h \in H_{\mathcal{L}}^1(G_Z, Ad\bar{\rho})$, $\phi \in H_{\mathcal{L}^\perp}^1(G_Z, Ad\bar{\rho}^*)$ and q be nice.*

1) *The injective inflation map*

$$H^1(G_Z, Ad\bar{\rho}) \rightarrow H^1(G_{Z \cup \{q\}}, Ad\bar{\rho})$$

has codimension 0 or 1. If $\text{III}^1(Ad\bar{\rho}^)|_{G_q} = 0$ then the codimension is 1.*

2) *If $h|_{G_q} \notin \tilde{\mathcal{L}}_q$ and $\phi|_{G_q} \neq 0$ then*

$$\dim H_{\mathcal{L}}^1(G_{Z \cup \{q\}}, Ad\bar{\rho}) = H_{\mathcal{L}}^1(G_Z, Ad\bar{\rho}) - 1, \dim H_{\mathcal{L}^\perp}^1(G_{Z \cup \{q\}}, Ad\bar{\rho}^*) = \dim H_{\mathcal{L}^\perp}^1(G_Z, Ad\bar{\rho}^*) - 1.$$

The proof of Proposition 17 is similar to that of Propostion 16 and not included. See [18] for the proof of 2).

Consider the deformation to the dual numbers given by

$$G_{\mathbb{Q}} \xrightarrow{\rho_n} GL_2(\mathcal{O}/(\pi^n)) \rightarrow GL_2(\mathcal{O}/(\pi^2)) \simeq GL_2(\mathbb{F}_{p^f}[U]/(U^2)).$$

The fullness assumption implies $f \neq 0$. As the determinant of the above composite representation is $\bar{\epsilon}$, $f \in H_{\bar{\epsilon}}^1(G_S, Ad^0 \bar{\rho}) \subset H_{\bar{\epsilon}}^1(G_S, Ad \bar{\rho})$, that is f lives in the trace zero cohomology. This is important as in the end f will span the tangent space of our arbitrary weight ordinary ring *and* the tangent space of its weight 2 quotient. By Corollary 8 we may take $\{\phi_1, \dots, \phi_s\}$ and $\{h_1, \dots, h_s, f\}$ as bases of $H_{\bar{\epsilon}^\perp}^1(G_S, Ad \bar{\rho}^*)$ and $H_{\bar{\epsilon}}^1(G_S, Ad \bar{\rho})$.

Lemma 18. *Let Q_i be the set of nice primes such that, for $q_i \in Q_i$*

- $\phi_i|_{G_{q_i}} \neq 0$ (equivalently, $\phi_{i, Ad^0 \bar{\rho}^*}|_{G_{q_i}} \neq 0$),
- $h_{i, Ad^0 \bar{\rho}}|_{G_{q_i}} \neq 0$ and $h_{i, sc}|_{G_{q_i}} = 0$,
- for $j \neq i$, $\phi_j, h_j|_{G_{q_i}} = 0$ and
- q_i is ρ_n -nice, that is $\rho_n(Fr_{q_i}) = \begin{pmatrix} q_i & 0 \\ 0 & 1 \end{pmatrix}$ where this element has order prime to p .

Then Q_i is nonempty.

Proof. It suffices to show the conditions above are independent Chebotarev conditions, that is the determine linearly disjoint extensions over $K := \mathbb{Q}(\bar{\rho})$, and thus can be simultaneously satisfied.

Each of the cohomology classes above, when restricted to the absolute Galois group of the field K , becomes an element of $Hom_{Gal(K/\mathbb{Q})}(G_K, M)$ for $M = Ad \bar{\rho}$ or $Ad \bar{\rho}^*$. For $M = Ad^0 \bar{\rho}$ or $Ad^0 \bar{\rho}^*$, the independence of the first three conditions has been established in [13] and [18]. The case of full adjoint cohomology results from these works and Proposition 15 as follows. Write $\phi_i = \phi_{i, Ad^0 \bar{\rho}^*} + \phi_{i, \mathbb{F}_{p^f}(1)}$ where $\phi_{i, Ad^0 \bar{\rho}^*} \in H^1(G_S, Ad^0 \bar{\rho}^*)$ and $\phi_{i, \mathbb{F}_{p^f}(1)} \in H^1(G_S, \mathbb{F}_{p^f}(1))$. We claim

the set $\{\phi_{1, Ad^0 \bar{\rho}^*}, \dots, \phi_{s, Ad^0 \bar{\rho}^*}\}$ is independent. Indeed, suppose $\sum_{j=1}^s a_j \phi_{j, Ad^0 \bar{\rho}^*} = 0$ is a dependence relation. Then

$$\sum_{j=1}^s a_j \phi_j = \sum_{j=1}^s a_j (\phi_{j, Ad^0 \bar{\rho}^*} + \phi_{j, \mathbb{F}_{p^f}(1)}) = \sum_{j=1}^s a_j \phi_{j, \mathbb{F}_{p^f}(1)} \in H_{\bar{\epsilon}^\perp}^1(G_S, Ad^0 \bar{\rho}^*) \cap H^1(G_S, \mu_p)$$

which is 0 by Proposition 15, a contradiction. Let L be the composite of the fields fixed by the kernels of $\phi_i|_{G_K}$. Then $Gal(L/K)$ contains, when viewed as a $\mathbb{F}_{p^f}[Gal(K/\mathbb{Q})]$ -module, s copies of $Ad^0 \bar{\rho}^*$ by [12], [18]. A similar argument gives that the composite of the fields fixed by the kernels of $h_i|_{G_K}$ and $f|_{G_K}$ contains $s+1$ copies of $Ad^0 \bar{\rho}$. This reduces the independence of the first three conditions to the same question with $Ad^0 \bar{\rho}$ and $Ad^0 \bar{\rho}^*$ cohomology where it is known.

The fourth condition is a complete splitting condition from K to L_n , the field fixed by the kernel of ρ_n . The Jordan-Hölder components of $Gal(K_n/K)$ are $\mathbb{F}_{p^f}[Gal(K/\mathbb{Q})]$ -submodules of $Ad \bar{\rho}$ that are either $Ad \bar{\rho}$ or $Ad^0 \bar{\rho}$. As the fields fixed by the kernels of the $\phi_i|_{G_K}$ give $Ad \bar{\rho}^*$ (or $Ad^0 \bar{\rho}^*$) extensions, these are linearly disjoint over K from L_n . The fields fixed by the kernels of the $h_i|_{G_K}$ give rise to $Ad \bar{\rho}$ (or $Ad^0 \bar{\rho}$) extensions of K . If the composite of these fields intersects L_n nontrivially then, as this intersection is abelian over K , the proof of Lemma 14 applied to ρ_n implies this composite contains $\text{Kernel}(f|_{G_K})$ and f is in the span of the trace zero parts of $\{h_{1, Ad^0 \bar{\rho}}, \dots, h_{s, Ad^0 \bar{\rho}}\}$. Proposition 15 then implies f is in the span of $\{h_1, \dots, h_s\}$, a contradiction. Thus the composite of the fields fixed by the h_i is linearly disjoint over K from L_n . \square

4.3. Proof of Theorem 11.

Lundell has proved the following

Theorem 19. (Lundell) *Let $\bar{\rho}$ be odd, ordinary, full and weight 2 with determinant ϵ . Suppose for a set T $\dim H_{\bar{\mathcal{L}}^\perp}^1(G_T, \text{Ad}\bar{\rho}^*) = 0$. Then $\dim H_{\bar{\mathcal{L}}}^1(G_T, \text{Ad}\bar{\rho}) = 1$ and the ordinary arbitrary weight deformation ring $R^{\text{ord}, T-\text{new}} \simeq W(\mathbb{F}_{p^f})[[U]] \simeq \mathbb{T}^{\text{ord}, T-\text{new}}$, the universal ordinary modular deformation ring associated to $\bar{\rho}$. Let $\{g\}$ be a basis of $H_{\bar{\mathcal{L}}}^1(G_T, \text{Ad}\bar{\rho})$. The weight 2 quotient $R_2^{\text{ord}, T-\text{new}} \simeq W(\mathbb{F}_{p^f})$ if and only if $g \notin H_{\bar{\mathcal{L}}}^1(G_T, \text{Ad}^0\bar{\rho})$, that is if and only if g does not belong to trace 0 cohomology.*

Remark: Using the ordinary deformation rings of Proposition 6, this theorem handles all possible $\bar{\rho} \mid_{G_p}$. It does not handle all possible $\rho_n \mid_{G_p}$ as 5) of the proposition only allows flat weight 2 deformations.

Proof of Theorem 11. Choose $q_i \in Q_i$ and set $Q = \{q_1, \dots, q_s\}$ and $T = S \cup Q$. Part 2) of Proposition 17 implies the Selmer and dual Selmer groups decrease in dimension by 1 for each q_i at which we allow ramification. Thus $H_{\bar{\mathcal{L}}^\perp}^1(G_T, \text{Ad}\bar{\rho}^*) = 0$ and $H_{\bar{\mathcal{L}}}^1(G_T, \text{Ad}\bar{\rho})$ is spanned by f so $R^{\text{ord}, T-\text{new}} \simeq W(\mathbb{F}_{p^f})[[U]]$. Since we assume $\bar{\rho}$ is modular and absolutely irreducible, [4] implies that $R^{\text{ord}, T-\text{new}}$ has characteristic zero points in all classical weights. Thus it is in fact a Hida family. By the fourth condition on the $q_i \in Q_i$ we see that that ρ_n arises as a point of $R_2^{\text{ord}, T-\text{new}}$, the weight 2 quotient of $R^{\text{ord}, T-\text{new}}$. \square

5. ρ_n LIFTS TO AN \mathcal{O} -VALUED WEIGHT 2 POINT

Suppose $\rho_n : G_{\mathbb{Q}} \rightarrow GL_2(\mathcal{O}/(\pi^n))$ is odd, ordinary, weight 2, modular, has full image, with determinant ϵ . We assume ρ_n is balanced. Throughout this section we assume for simplicity that \mathcal{O} is totally ramified.

Suppose also that $\rho_n \mid_{G_v} \in \mathcal{C}_v$ for all $v \in S$. Recall that for $v = p$ we require that if $\bar{\rho} = \begin{pmatrix} \bar{\epsilon} & 0 \\ 0 & 1 \end{pmatrix}$ then \mathcal{C}_p is taken to be the flat deformations, so we *must* assume $\rho_n \mid_{G_p}$ is flat.

If $\bar{\rho} \mid_{G_p} = \begin{pmatrix} \bar{\epsilon} & * \\ 0 & 1 \end{pmatrix}$ is flat and ρ_n is flat (but not semistable), we take \mathcal{C}_p to be the flat deformations. If ρ_n is not flat but semistable, we take \mathcal{C}_p to be the semistable deformations. If ρ_n is both flat and semistable, we take \mathcal{C}_p to be the flat or semistable deformations however we please. The universal representations corresponding to the deformation rings considered in Theorem 20 are locally at p of type \mathcal{C}_p .

The goal of this section is to prove Theorem 20.

Theorem 20. *There exists a finite set of primes $T_2 \supseteq S$ such that $R^{\text{ord}, T_2-\text{new}} \simeq W(\mathbb{F}_{p^f})[[U]]$ and there are maps*

$$W(\mathbb{F}_{p^f})[[U]] \simeq R^{\text{ord}, T_2-\text{new}} \rightarrow R_2^{\text{ord}, T_2-\text{new}} \rightarrow \mathcal{O} \rightarrow \mathcal{O}/(\pi^n)$$

inducing ρ_n . The map $R_2^{\text{ord}, T_2-\text{new}} \rightarrow \mathcal{O}$ is an isomorphism for $\bar{\rho}$ in cases 1), 2) and 4) of Proposition 6. For the exceptional $\bar{\rho}$ described above (cases 3) and 5) of Proposition 6), we have that \mathcal{O} is isomorphic to either the semistable or flat quotient of the weight 2 deformation ring, according to how we have chosen \mathcal{C}_p .

In this section we will study various universal ordinary rings $R^{\text{ord}, ?-\text{new}}$ associated to $\bar{\rho}$ and their weight 2 quotients $R_2^{\text{ord}, ?-\text{new}}$. If we are in an exceptional case of Theorem 20 we need a quotient of the weight 2 ring having property $*$ $\in \{\text{flat}, \text{semistable}\}$. We will use the notation $R_{2*}^{\text{ord}, ?-\text{new}}$ to indicate this quotient of the full weight 2 ring. In cases 1), 2) and 4) of Proposition 6 $R_2^{\text{ord}, ?-\text{new}} \simeq R_{2*}^{\text{ord}, ?-\text{new}}$.

The weight 2 quotient of any $R^{ord, X-new} \simeq W(\mathbb{F}_{p^f})[[U]]$ is formed by quotienting out $R^{ord, X-new}$ by a relation $w_2(U)$ that fixes the determinant of the X -new deformation to be the cyclotomic character.

Lemma 21. *The modularity of $\bar{\rho}$ implies we may assume $w_2(U) \in W(\mathbb{F}_{p^f})[[U]]$ is a distinguished polynomial.*

Proof. Set $j(U)$ to be the determinant of our ordinary representation evaluated at a topological generator of the Galois group of the cyclotomic \mathbb{Z}_p -extension of \mathbb{Q} so that at a weight t point $U_t \in \mathfrak{m}_{\mathbb{Z}_p}$, $j(U_t) = (1+p)^{t-1}$. Writing $j(U) = 1+p+w_2(U)$, the weight 2 quotient is $W(\mathbb{F}_{p^f})[[U]]/(w_2(U))$. The Weierstrass preparation theorem implies $w_2(U) = p^t v_2(U)u(U)$ where v_2 is a distinguished polynomial and $u(U)$ is a unit. We need to prove $t = 0$ so suppose $t \geq 1$.

By [4] $R^{ord, X-new} \simeq W(\mathbb{F}_{p^f})[[U]]$ has a point of each classical weight $k \geq 2$. The weight 3 quotient is formed imposing the relation $j(U) = (1+p)^2$, that is quotienting out by

$$j(U) - (1+p)^2 = w_2(U) - p - p^2 = p^t v_2(U)u(U) - p - p^2 = p[-1 - p + p^{t-1} v_2(U)u(U)].$$

As there is at least one weight 2 point in this Hida family, $v_2(U)$ has positive degree so the rightmost quantity above is p times a unit and is never 0 for any choice of U . Thus if $t \geq 1$ there are no weight 3 points, a contradiction. \square

The roots of $w_2(U)$ will give rise to all weight 2 deformations of $\bar{\rho}$. In cases 3) and 5) of Proposition 6, we have $w_{2,fl}(U), w_{2,ss}(U) | w_2(U)$ where the roots of $w_{2,*}$ give rise to weight 2 points with the appropriate local at p property. *A priori* $w_{2,fl}$ and $w_{2,ss}$ could share a root, but geometricity and the Weil bounds imply this is not the case, though we do not need this for our purposes. Henceforth we will write $R/(w_{2,*}(U))$ to indicate the weight two quotient with which we are dealing (the full weight 2 quotient except in cases 3) and 5) of Proposition 6) and will control to be \mathcal{O} .

5.1. Recollection of earlier work in [14]. Let T be as in Theorem 11. A key technical ingredient in this section is the main lifting result of [14], which in turn builds on [8]. The point of [14] was to build a pathological Galois representation by removing all obstructions to deformation problems. Here we repeat this procedure for a *finite number of steps*, but then we *introduce* an obstruction later to force $R_{2,*}^{ord, T_2-new}$ to be ‘close to’ a specified ring. This closeness will allow us to choose $R_{2,*}^{ord, T_2-new}$ to be isomorphic to a given totally ramified extension of $W(\mathbb{F}_{p^f})$.

We recall some of the key ingredients of [14]. In [14] the only the fields \mathbb{F}_p for $p \geq 5$ were used. Recall that here we consider \mathbb{F}_{p^f} with $q = p^f$ and $p \geq 3$. First consider the hypotheses of section 4 of [14]:

- Fullness of the image of $\bar{\rho}$ we assume here.
- Triviality of

$$\text{III}_T^1(Ad^0 \bar{\rho}^*) := \text{Kernel} \left(H^1(G_T, Ad^0 \bar{\rho}^*) \rightarrow \bigoplus_{v \in T} H^1(G_v, Ad^0 \bar{\rho}^*) \right)$$

can be realised as follows. Note that for $T \subset Z$, $\text{III}_Z^1(Ad^0 \bar{\rho}^*) \subset \text{III}_T^1(Ad^0 \bar{\rho}^*)$. Then let $\{\theta_1, \dots, \theta_t\}$ be a basis for $\text{III}_T^1(Ad^0 \bar{\rho}^*)$ and choose nice primes q_i such that

- q_i is ρ_n -nice. Recall this is a complete splitting condition on q_i in $\text{Gal}(L_n/\mathbb{Q}(\bar{\rho}, \mu_p))$,
- $H^1(G_T, Ad\bar{\rho})|_{G_{q_i}} = 0$ and
- $\theta_i|_{G_{q_i}} \neq 0$ and $j \neq i \implies \theta_j|_{G_{q_i}} = 0$.

Replacing T by $T \cup \{q_1, \dots, q_t\}$ (which we rename T) gives $\text{III}_T^1(Ad^0 \bar{\rho}^*) = 0$.

- The third hypothesis of section 4 of [14] was that the local deformations be specified uniquely. This was equivalent to specifying $W(\mathbb{F}_{p^f})$ as our smooth quotient of each local deformation ring. We simply ignore that here and use \mathcal{C}_v as our (weight 2) set of local points as usual.

Suppose now we have an ordinary weight 2 deformation of $\bar{\rho}$, $\rho_R : G_Z \rightarrow GL_2(R)$ where R is a finite complete local Noetherian ring with residue field \mathbb{F}_{p^f} and $\rho_R|_{G_v} \in \mathcal{C}_v$ for all $v \in Z$. Let R_1 be another such ring and

$$R_2 \twoheadrightarrow R_1 \xrightarrow{\delta} R$$

be surjections with $\text{Ker}(\delta) = (b)$, a principal ideal isomorphic to \mathbb{F}_{p^f} . It is natural to ask whether ρ_R deforms to a $\rho_{R_1} : G_Z \rightarrow GL_2(R_1)$ of weight 2. The obstruction lies in $H^2(G_Z, \text{Ad}^0 \bar{\rho})$. As $\text{III}_T^1(\text{Ad}^0 \bar{\rho}^*)$ is dual to $\text{III}_T^2(\text{Ad}^0 \bar{\rho})$, the second bullet point above implies this obstruction is realised locally. But as $\rho_R|_{G_v} \in \mathcal{C}_v$ for all $v \in Z$ and the \mathcal{C}_v represent the points of a smooth ring, there are no local obstructions and ρ_{R_1} exists. It may be, however, that there are $v_0 \in Z$ with $\rho_{R_1}|_{G_{v_0}} \notin \mathcal{C}_{v_0}$. In this case deforming to R_2 may not be possible. The smoothness of the local deformation rings implies that $\rho_R|_{G_v}$ has a deformation to $GL_2(R_1)$ arising from \mathcal{C}_v for all $v \in Z$. The obstruction to deforming $\rho_{R_1}|_{G_{v_0}}$ to R_2 can be removed by a cohomology class $z_{v_0} \in H^1(G_{v_0}, \text{Ad}^0 \bar{\rho})$. We call the collection $(z_v)_{v \in Z}$ the *local condition problem* for ρ_{R_1} . Proposition 3.4 of [14] shows that there exists a ρ_{R_1} -nice prime q , and $h \in H^1(G_{Z \cup \{q\}}, \text{Ad}^0 \bar{\rho})$ that solves the local condition problem above, that is $(I + h)\rho_{R_1}|_{G_v} \in \mathcal{C}_v$ for all $v \in Z$. The difficulty is that we cannot guarantee $(I + h)\rho_{R_1}|_{G_q} \in \mathcal{C}_q$. If this fails for all ρ_{R_1} -nice primes, Proposition 3.6 of [14] shows how to add two nice primes q_1 and q_2 to Z and find a cohomology class $h \in H^1(G_{Z \cup \{q_1, q_2\}}, \text{Ad}^0 \bar{\rho})$ such that $(I + h)\rho_{R_1}|_{G_v} \in \mathcal{C}_v$ for all $v \in Z \cup \{q_1, q_2\}$.

5.2. Strategy of the proof of Theorem 20. We use the integer N throughout this section to denote a large natural number. This largeness will depend only on $\rho_n : G_{\mathbb{Q}} \rightarrow GL_2(\mathcal{O}/(\pi^n))$. We explain the strategy, which gives an indication of how N is chosen. The first step is to construct a deformation problem where the arbitrary weight ordinary deformation ring will be $W(\mathbb{F}_{p^f})[[U]]$ and its weight 2 quotient will surject onto $W(\mathbb{F}_{p^f})[[U]]/(p^N, U^{Ne})$ which in turn will surject onto $\mathcal{O}/(\pi^n)$ and give rise to ρ_n . So

$$W(\mathbb{F}_{p^f})[[U]] \simeq R^{ord} \twoheadrightarrow R_{2*}^{ord} = W(\mathbb{F}_{p^f})[[U]]/(w_{2*,N}(U)) \twoheadrightarrow W(\mathbb{F}_{p^f})[[U]]/(p^N, U^{Ne}) \twoheadrightarrow \mathcal{O}/(\pi^n).$$

If in this composite U maps to an element of $\mathcal{O}/(\pi^n)$ whose various lifts to \mathcal{O} have valuation greater than $1/e$ then the deformation to $\mathcal{O}/(\pi^2)$ would be trivial, contradicting the fullness of ρ_n . Thus U maps to an element whose lifts to \mathcal{O} are uniformisers. After multiplying by a unit, we may assume $U \mapsto \pi$. Our strategy is to then alter the problem by allowing more ramification so that $w_{2*,N}(U)$ is exactly of degree e and ‘close to’ $g_{\pi}(U)$, the minimal polynomial of π over $W(\mathbb{F}_{p^f})$. The choice of N will depend on $|\pi^n|$ and the Krasner bound on the distances between roots of $g_{\pi}(U)$. Furthermore $w_{2*,N}$ has a root and has a root $y_{N,1}$ such that the deformation given by $U \mapsto y_{N,1}$ gives rise to ρ_n . Thus ρ_n will have a weight 2 characteristic 0 lifting.

5.3. Weight 2 deformation rings that are large. In this section we will construct large weight 2 deformation rings that give rise to ρ_n . The technical hypotheses on $\rho_n|_{G_p}$ in the introduction arise here.

Proposition 22. *For any integer N , there exists a set $X_N \supseteq T$ such that*

$$R_{2*}^{ord, X_N - \text{new}} \twoheadrightarrow W(\mathbb{F}_{p^f})[[U]]/(p^N, U^{Ne}) \twoheadrightarrow \mathcal{O}/(\pi^n).$$

Proof. Theorem 1.1 of [14], based on techniques of [8], gives examples of weight 2 deformation rings that are arbitrarily large and ramified at infinitely many primes. It is proved by taking an inverse limit of certain finite cardinality quotients of deformation rings that satisfy a specified property at $v \in S_0$, namely the local representation at G_v is (the reduction of) a specific deformation of $\bar{\rho}|_{G_v}$ to \mathbb{Z}_p . While the ring $W(\mathbb{F}_{p^f})[[U]]/(p^N, U^{Ne})$ is not explicitly included there, the techniques apply. The deformation of Theorem 11 to $\mathcal{O}/(\pi^n)$ factors

$$(5.1) \quad W(\mathbb{F}_{p^f})[[U]] = R^{ord, T - \text{new}} \twoheadrightarrow R_{2*}^{ord, T - \text{new}} = W(\mathbb{F}_{p^f})[[U]]/(w_{2*}(U)) \twoheadrightarrow \mathcal{O}/(\pi^n)$$

and $U \mapsto \pi$ in the composite. Let $g_\pi(U)$ be the minimal polynomial of π over $W(\mathbb{F}_{p^f})$. Observe that both $g_\pi(U)$ and U^n are in the kernel of the composite map. As $U \mapsto \pi$ gives an isomorphism $W(\mathbb{F}_{p^f})[[U]]/(g_\pi(U), U^n) \simeq \mathcal{O}/(\pi^n)$ the kernel of (5.1) is $(g_\pi(U), U^n)$ so $w_{2*}(U) \in (g_\pi(U), U^n)$. For $N \geq n$ note $p^N, U^{Ne} \in (g_\pi(U), U^n)$ as they both map to 0 in (5.1).

We will construct a ring $R = W(\mathbb{F}_{p^f})[[U]]/I$ (not yet a deformation ring!) that surjects onto $W(\mathbb{F}_{p^f})[[U]]/(p^N, U^{Ne})$ and from there onto $W(\mathbb{F}_{p^f})[[U]]/(g_\pi(U), U^n) \simeq \mathcal{O}/(\pi^n)$. We'll build R as a series of small extensions of $W(\mathbb{F}_{p^f})[[U]]/(g_\pi(U), U^n)$ and then invoke Proposition 3.6 of [14] to realise all of these rings as quotients of weight 2 deformation rings. Consider the map

$$W(\mathbb{F}_{p^f})[[U]]/(pg_\pi(U), Ug_\pi(U), U^n) \rightarrow W(\mathbb{F}_{p^f})[[U]]/(g_\pi(U), U^n).$$

The kernel is just the ideal $(g_\pi(U))$ and this is killed by (p, U) , the maximal ideal of $W(\mathbb{F}_{p^f})[[U]]$, so the extension is small. Similarly, the map

$$W(\mathbb{F}_{p^f})[[U]]/(pg_\pi(U), Ug_\pi(U), pU^n, U^{n+1}) \rightarrow W(\mathbb{F}_{p^f})[[U]]/(pg_\pi(U), Ug_\pi(U), U^n)$$

has kernel (U^n) and this is also killed by (p, U) . Repeat this process (with more and more elements in our ideal) until all the generators are of the form $p^r U^s g_\pi(U)$ or $p^r U^s U^n$ where $r+s = N+Ne$. Let I be this ideal of relations. In each relation either $r \geq N$ or $s \geq Ne$ so $I \subset (p^N, U^{Ne}) \subset (g_\pi(U), U^n)$. Then

$$W(\mathbb{F}_{p^f})[[U]]/I \twoheadrightarrow \cdots \twoheadrightarrow W(\mathbb{F}_{p^f})[[U]]/(g_\pi(U), U^n) \simeq \mathcal{O}/(\pi^n)$$

is a series of small extensions.

Now we'll use [14] to deform ρ_n to each small extension, perhaps allowing ramification at one or two nice primes at each step. If we can deform ρ_n all the way to $W(\mathbb{F}_{p^f})[[U]]/I$ without allowing more ramification, then we are done. If not, there is a first place at which the small deformation problem is obstructed. This is not at ρ_n as $\text{III}_T^2(Ad^0 \bar{\rho}) = 0$ (being dual to $\text{III}_T^1(Ad^0 \bar{\rho}^*)$) and the local deformation problems ρ_n are assumed unobstructed. The smoothness of the chosen quotients of the local deformation rings implies there are local cohomology classes $(h_v)_{v \in T}$ that 'unobstruct' each of the given local deformation problems. Proposition 3.4 of [14] implies that with one nice prime q the local deformation problems at $v \in T$ can be 'unobstructed' by a global class in $H^1(G_{T \cup \{q\}}, Ad^0 \bar{\rho})$, but possibly this class introduces an obstruction at q . If all nice primes introduce such an obstruction, the rest of section 3 of [14] shows how to allow ramification at two nice primes $\{q_1, q_2\}$ so that the obstruction introduced at these primes cancel one another. Then we deform and move on to the next small extension. Set X_N to be the final set of nice primes. \square

The primes q used in Proposition 22 were ρ_n -nice so $f|_{G_q} = 0$ and $f \in H_{\mathcal{L}}^1(G_{X_N}, Ad^0 \bar{\rho}) \subset H_{\mathcal{L}}^1(G_{X_N}, Ad \bar{\rho})$ but this last space could have dimension > 1 so the first step in our strategy is not yet complete. Lemma 23 remedies this.

Lemma 23. *There exists a set Y_N containing X_N of Proposition 22 such that $\dim H_{\mathcal{L}^\perp}^1(G_{Y_N}, Ad \bar{\rho}^*) = 0$ and $\dim H_{\mathcal{L}}^1(G_{Y_N}, Ad \bar{\rho}) = 1$ so $R^{ord, Y_N - new} \simeq W(\mathbb{F}_{p^f})[[U]]$. Furthermore*

$$(5.2) \quad W(\mathbb{F}_{p^f})[[U]] \simeq R^{ord, Y_N - new} \twoheadrightarrow R_{2*}^{ord, Y_N - new} = W(\mathbb{F}_{p^f})[[U]]/(w_{2*, N}(U)) \twoheadrightarrow W(\mathbb{F}_{p^f})[[U]]/(p^N, U^{Ne}) \twoheadrightarrow \mathcal{O}/(\pi^n).$$

Proof. The proof is similar to that of Lemma 18. Let $\rho_N : G_{X_N} \rightarrow GL_2(W(\mathbb{F}_{p^f})[[U]]/(p^N, U^{Ne}))$ be the deformation of Proposition 22. We will need that the image of ρ_N is full. While this is easy to prove for $p > 3$ as then (5.3) is nonsplit, the proof below works for $p = 3$ and exploits ordinarity.

The ring $W(\mathbb{F}_{p^f})[[U]]/(p^N, U^{Ne})$ has the quotient $W(\mathbb{F}_{p^f})/(p^2)$ and there is the extension

$$(5.3) \quad 1 \rightarrow Ad^0 \bar{\rho} \rightarrow GL_2(W(\mathbb{F}_{p^f})/(p^2)) \rightarrow GL_2(\mathbb{F}_{p^f}) \rightarrow 1.$$

When restricted to G_p the deformation

$$\gamma : G_{Y_N} \rightarrow GL_2(W(\mathbb{F}_{p^f})/(p^2))$$

is of the form $\begin{pmatrix} \epsilon\psi & * \\ 0 & \psi^{-1} \end{pmatrix}$ where the ψ is unramified. Thus the image of γ contains a lifting of $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{F}_{p^f})$ to an element of order $2p$ in $GL_2(W(\mathbb{F}_{p^f})/(p^2))$ so the image is full. The cohomology class f also gives rise to an extension with the same short exact sequence as above corresponding to the deformation to the dual numbers. We show the $Ad^0\bar{\rho}$ parts of these extensions are different by considering the images of G_p . The class $f \in H^1(G_{Y_N}, Ad^0\bar{\rho})$ belongs to the trace zero cohomology. When comparing the field extension it cuts out over $\mathbb{Q}(\bar{\rho})$ to that of γ , it suffices to twist γ by a character ϕ so $\det(\gamma \otimes \phi) = \tilde{\epsilon}$, the Teichmüller lift of ϵ . One easily sees that the image of inertia at p in $((\gamma \otimes \phi)(G_p))^{ab}$ is at least of order p while inertia at p has image of order $p-1$ in the corresponding abelianisation of the lift to the dual numbers arising from f . Thus the $Ad^0\bar{\rho}$ extensions are distinct and the image of $\rho_{R_2^{ord, Y_N - new}} \bmod (p, U)^2$ is full. Lemma 12 implies $\rho_{R_2^{ord, Y_N - new}} \bmod (p^N, U^{Ne})$ has full image.

Now take $\{\phi_1, \dots, \phi_s\}$ and $\{h_1, \dots, h_s, f\}$ as bases for $H_{\tilde{\mathcal{L}}^\perp}^1(G_{X_N}, Ad\bar{\rho}^*)$ and $H_{\tilde{\mathcal{L}}}^1(G_{X_N}, Ad\bar{\rho})$. As before $f \in H_{\tilde{\mathcal{L}}}^1(G_{X_N}, Ad^0\bar{\rho})$ is the cohomology class arising from $\rho_n \bmod (\pi^2)$. Let Q_i be the Chebotarev set of primes q_i satisfying

- $\phi_i|_{G_{q_i}} \neq 0$ (equivalently $\phi_{i, Ad^0\bar{\rho}^*} \neq 0$),
- $h_i|_{G_{q_i}} \notin \tilde{\mathcal{L}}_{q_i}$ (equivalently $h_{i, Ad^0\bar{\rho}}|_{G_{q_i}} \neq 0$) and $h_{i, sc}|_{G_{q_i}} = 0$,
- for $j \neq i$, $\phi_j, h_j|_{G_{q_i}} = 0$ and
- q_i is ρ_N -nice, that is $\rho_N(Fr_{q_i}) = \begin{pmatrix} q_i & 0 \\ 0 & 1 \end{pmatrix}$ where this element has order prime to p .

Setting

$$\Gamma = \{A \in \text{Image}(\rho_N) \mid A \equiv I \bmod (p, U)\}$$

and using the fullness of ρ_N established above, one can easily adapt the proof of Lemma 14 to show $\dim H^1(\text{Image}(\rho_N), Ad^0\bar{\rho}) = 1$. One then modifies Lemma 18 to show the above bullet points are independent Chebotarev conditions.

Let $q_i \in Q_i$ and set $Y_N = X_N \cup \{q_1, \dots, q_s\}$, Then by part 2) of Propostion 17 $\dim H_{\tilde{\mathcal{L}}^\perp}^1(G_{Y_N}, Ad\bar{\rho}^*) = 0$ and $\dim H_{\tilde{\mathcal{L}}}^1(G_{Y_N}, Ad\bar{\rho}) = 1$ and this last group has basis $\{f\}$. The ordinary ring $R^{ord, Y_N - new} \simeq W(\mathbb{F}_{p^f})[[U]]$. The fourth condition on the Q_i implies that the deformation ρ_N arises from the weight 2 quotient of $R^{ord, Y_N - new}$ so $R_2^{ord, Y_N - new} \twoheadrightarrow W(\mathbb{F}_{p^f})[[U]]/(p^N, U^{Ne})$. \square

The second surjection of (5.2) implies $w_{2^*, N}(U) \in (p^N, U^{Ne})$ and Lemma 21 implies $w_{2, N}(U)$ is a distinguished polynomial. It thus has degree at least Ne .

5.4. Cutting down the size of weight 2 deformation rings via local obstructions. Our next step is to add more nice primes of ramification so that the new weight 2 ordinary ring is a quotient of $W(\mathbb{F}_{p^f})[[U]]$ by a polynomial $v_{2, N}(U)$ of degree *exactly* e . Furthermore, $v_{2, N}(U)$ will have a root $y_{N, 1}$ such that $U \mapsto y_{N, 1}$ gives rise to ρ_n .

Let C be a positive number smaller than both $|\pi^n|$ and the minimum of half the distances between any pairs of roots of g_π , its Krasner bound.

Recall that $U \mapsto \pi$ in

$$G_{Y_N} \xrightarrow{\rho_{R^{ord, Y_N - new}}} GL_2(W(\mathbb{F}_{p^f})[[U]]) \rightarrow GL_2(W(\mathbb{F}_{p^f})[[U]]/(p^N, U^{Ne})) \rightarrow GL_2(\mathcal{O}/(\pi^n)).$$

Denote by $\rho_{g_\pi, N}$ the deformation

$$\begin{aligned} G_{Y_N} &\rightarrow GL_2\left(R^{ord, Y_N - new} = W(\mathbb{F}_{p^f})[[U]]\right) \rightarrow GL_2\left(R_2^{ord, Y_N - new}\right) \\ &\rightarrow GL_2\left(W(\mathbb{F}_{p^f})[[U]]/(p^N, U^{Ne})\right) \rightarrow GL_2\left(W(\mathbb{F}_{p^f})[[U]]/(p^N, g_\pi(U), U^{Ne})\right). \end{aligned}$$

Let $\rho_{p,k}$ be the reduction of the deformation $G_{Y_N} \xrightarrow{\rho_{R^{ord}, Y_N - new}} GL_2(W(\mathbb{F}_{p^f})[[U]]/(p^N, U^{Ne})) \bmod (p, U^k)$.

Note

- $\dim H_{\tilde{\mathcal{L}}}^1(G_{Y_N}, Ad\bar{\rho}) = 1$ and this space has basis $\{f\}$,
- $\dim H_{\tilde{\mathcal{L}}}^1(G_{Y_N}, Ad^0\bar{\rho}) = 1$ and this space has basis $\{f\}$,
- $\dim H_{\tilde{\mathcal{L}}^\perp}^1(G_{Y_N}, Ad\bar{\rho}^*) = 0$ and
- $\dim H_{\tilde{\mathcal{L}}^\perp}^1(G_{Y_N}, Ad^0\bar{\rho}^*) = 1$ and this space has some basis, say $\{\phi\}$.

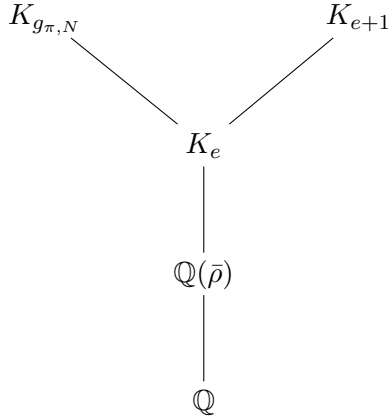
Let Q be the set of primes q satisfying

- $H^1(G_{Y_N}, Ad^0\bar{\rho})|_{G_q} = 0$,
- q is $\rho_{g_{\pi,N}}$ -nice.
- $\phi|_{G_q} \neq 0$,
- $\rho_{p,e+1}(Fr_q) = \begin{pmatrix} q(1+U^e) & 0 \\ 0 & 1-U^e \end{pmatrix}$.

Proposition 24. *The Chebotarev conditions defining Q above are independent for $p \geq 3$.*

Proof. The first two conditions are complete splitting conditions in fields above $\mathbb{Q}(\bar{\rho})$ and can therefore be satisfied simultaneously. They are both $Ad\bar{\rho}$ conditions and thus independent of the third condition, an $Ad\bar{\rho}^*$ condition. It remains to show the independence of the fourth condition from the previous three. Since it is a succession of $Ad\bar{\rho}$ conditions, we only have to check independence with the first two conditions and, since $e > 1$, independence with the first condition follows from Lemma 14.

Finally we show the independence of the fourth and second conditions. Let $K_{g_{\pi,N}}$, K_e and K_{e+1} be the fields fixed by the kernels of $\rho_{g_{\pi,N}}$, $\rho_{p,e}$ and $\rho_{p,e+1}$ respectively.



As $g_\pi(U)$ is distinguished of degree e , $K_{g_{\pi,N}} \supset K_e$. We'll show $K_{g_{\pi,N}} \not\supset K_{e+1}$. The same argument given in the proof of Lemma 23 implies $\rho_{R_{2*}^{ord}, Y_N - new} \bmod (p, U)^2$ has full image. Lemma 12 then implies the image of $\rho_{R_{2*}^{ord}, Y_N - new} \bmod (p^N, U^{Ne})$ contains $\begin{pmatrix} 1+g_\pi(U) & 0 \\ 0 & (1+g_\pi(U))^{-1} \end{pmatrix}$. When we reduce mod $(p^N, g_\pi(U), U^{Ne})$ to $\rho_{g_{\pi,N}}$ this element becomes trivial. But when we reduce mod (p, U^{e+1}) to $\rho_{p,e+1}$, bearing in mind that $g_\pi(U) \equiv U^e \bmod p$, the image is $\begin{pmatrix} 1+U^e & 0 \\ 0 & 1-U^e \end{pmatrix}$. So $K_{g_{\pi,N}} \not\supset K_{e+1}$. Thus $K_{g_{\pi,N}}$ and K_{e+1} are linearly disjoint over K_e . The second condition is a complete splitting condition in $K_{g_{\pi,N}}$ while the fourth is a complete splitting condition in K_e , but **not** in K_{e+1} . \square

5.5. Proof of Theorem 20. Choose $q_1 \in Q$. Part 4) of Proposition 16, using the first and third bullet points on q_1 , implies

$$\dim H_{\mathcal{L}}^1(G_{Y_N \cup \{q_1\}}, Ad^0 \bar{\rho}) = 1 = \dim H_{\mathcal{L}^\perp}^1(G_{Y_N \cup \{q_1\}}, Ad^0 \bar{\rho}^*)$$

and $H_{\mathcal{L}}^1(G_{Y_N \cup \{q_1\}}, Ad^0 \bar{\rho})$ is spanned by $\{f\}$ and $H_{\mathcal{L}^\perp}^1(G_{Y_N \cup \{q_1\}}, Ad^0 \bar{\rho}^*)$ is spanned by some $\{\tilde{\phi}\}$ ramified at q_1 . Thus $R_{2*}^{ord, Y_N \cup \{q_1\} - new}$ is a quotient of $W(\mathbb{F}_{p^f})[[U]]$ with one dimensional tangent space.

By part 1) of Proposition 17 there are two possibilities:

- $\dim H_{\mathcal{L}}^1(G_{Y_N \cup \{q_1\}}, Ad \bar{\rho}) = 1$ or
- $\dim H_{\mathcal{L}}^1(G_{Y_N \cup \{q_1\}}, Ad \bar{\rho}) = 2$.

In the first case, $R^{ord, Y_N \cup \{q_1\} - new} \simeq W(\mathbb{F}_{p^f})[[U]]$ and its weight 2^* quotient is formed by quotienting by the one determinant relation $v_{2*, N}(U)$ which we can assume is a distinguished polynomial by Lemma 21. In the latter case it is possible that $R_{2*}^{ord, Y_N \cup \{q_1\} - new}$ is a quotient of $W(\mathbb{F}_{p^f})[[U]]$ by either multiple relations or that it might not be finite and flat over $W(\mathbb{F}_{p^f})$. We will deal with this case by adding another prime of ramification.

While each $q_1 \in Q$ puts us in one of the two cases above, it is an open (and difficult!) question if both cases can occur. Part of the length of the argument below is because of this. That we do not know whether we have to allow ramification at one or two nice primes to remove obstructions to deformation problems here and in [8] is the same phenomenon.

5.5.1. Case 1. In the first case we have deformations associated to the ring homomorphisms

$$W(\mathbb{F}_{p^f})[[U]] \simeq R^{ord, Y_N \cup \{q_1\} - new} \twoheadrightarrow R_{2*}^{ord, Y_N \cup \{q_1\} - new} \simeq W(\mathbb{F}_{p^f})[[U]] / (v_{2*, N}(U)) \twoheadrightarrow W(\mathbb{F}_{p^f})[[U]] / (p^N, g_\pi(U), U^{Ne})$$

The last surjection above implies $v_{2*, N}(U) \in (p^N, g_\pi(U), U^{Ne})$ so its degree is at least e . We claim it is exactly e .

If the degree is greater than e , then $R_{2*}^{ord, Y_N \cup \{q_1\} - new} \twoheadrightarrow \mathbb{F}_{p^f}[[U]] / (U^{e+1})$. Call the corresponding deformation α and let β be the deformation induced by the composite

$$R_2^{ord, Y_N - new} \twoheadrightarrow W(\mathbb{F}_{p^f})[[U]] / (p^N, U^{Ne}) \twoheadrightarrow \mathbb{F}_{p^f}[[U]] / (U^{Ne}) \twoheadrightarrow \mathbb{F}_{p^f}[[U]] / (U^{e+1}).$$

Note $\alpha|_{G_{q_1}} \in \mathcal{C}_{q_1}$ and

$$\beta(Fr_{q_1}) = \begin{pmatrix} q_1(1 + U^e) & 0 \\ 0 & 1 - U^e \end{pmatrix} \implies \beta|_{G_{q_1}} \notin \mathcal{C}_{q_1},$$

that is β is not Steinberg at q_1 . As both α and β are deformations of $\rho_{g_\pi, N} \bmod p$ to $GL_2(\mathbb{F}_{p^f}[[U]] / (U^{e+1}))$ they differ by a 1-cohomology class $k \in H^1(G_{Y_N \cup \{q_1\}}, Ad^0 \bar{\rho})$, that is

$$\alpha = (I + U^e k) \beta.$$

If k is unramified at q_1 , then k inflates from $H^1(G_{Y_N}, Ad^0 \bar{\rho})$. But q_1 was chosen so $H^1(G_{Y_N}, Ad^0 \bar{\rho})|_{G_{q_1}} = 0$. Thus k cannot change the local at q_1 deformation where $\beta(Fr_{q_1}) = \begin{pmatrix} q_1(1 + U^e) & 0 \\ 0 & 1 - U^e \end{pmatrix}$ to one in \mathcal{C}_{q_1} . So k is ramified at q_1 . But we chose q_1 such that $\phi|_{G_{q_1}} \neq 0$ where ϕ spanned $H_{\mathcal{L}^\perp}^1(G_{Y_N}, Ad^0 \bar{\rho}^*)$. Parts 1) and 2) of Proposition 16 then imply the map

$$H^1(G_{Y_N \cup \{q_1\}}, Ad^0 \bar{\rho}) \rightarrow \oplus_{v \in Y_N} \frac{H^1(G_v, Ad^0 \bar{\rho})}{\mathcal{L}_v}$$

has image one dimension larger than the map

$$H^1(G_{Y_N}, Ad^0 \bar{\rho}) \rightarrow \oplus_{v \in Y_N} \frac{H^1(G_v, Ad^0 \bar{\rho})}{\mathcal{L}_v}.$$

For all $v \in Y_N$ we have $\alpha|_{G_v}$ belongs to our deformable class \mathcal{C}_v as does $\beta|_{G_v}$. But for at least one v we have $k_v|_{G_v} \notin \mathcal{L}_v$ so $\alpha|_{G_v} = (I + U^e k)\beta|_{G_v} \notin \mathcal{C}_v$, a contradiction. Thus k can be neither ramified nor unramified at q_1 . This contradiction implies $v_{2^*,N}(U)$ has degree $e \bmod p$.

Recall $v_{2^*,N}(U) \in (p^N, g_\pi(U), U^{Ne})$ so

$$(5.4) \quad v_{2^*,N}(U) = a(U)p^N + b(U)g_\pi(U) + c(U)U^{Ne}.$$

Recall that $g_\pi(U)$ is the minimal polynomial over $W(\mathbb{F}_{p^f})$ of π and that its roots are distinct. Since both $g_\pi(U)$ and $v_{2^*,N}(U)$ are degree e , $b(U)$ is a unit. Let $\{y_{N,1}, y_{N,2}, \dots, y_{N,e}\}$ be the roots of $v_{2^*,N}(U)$. As $v_{2^*,N}(U)$ is distinguished of degree e , $v_p(y_{N,i}) \geq 1/e$. Observe

$$0 = v_{2^*,N}(y_{N,i}) = p^N a(y_{N,i}) + b(y_{N,i})g_\pi(y_{N,i}) + c(y_{N,i})y_{N,i}^{Ne}.$$

The outside terms on the right have valuation at least N and $b(y_{N,i})$ is a unit, so $v_p(g_\pi(y_{N,i})) \geq N$. Thus $y_{N,i}$ is very close to a root of $g_\pi(U)$. For N large enough, this closeness is closer than the common Krasner bound C on the roots of $g_\pi(U)$. We claim each $y_{N,i}$ is close to a different root of $g_\pi(U)$. If this were false then a root of g_π would be missed, that is there would be a root x_0 of $g_\pi(U)$ with $|x_0 - y_{N,i}| > C$ for all i . As $v_{2^*,N}(U) = \prod (U - y_{N,i})$, we would have $|v_{2^*,N}(x_0)| > C^e$. Evaluating (5.4) at x_0 gives $|v_{2^*,N}(x_0)| < p^{-N}$, a contradiction for large N so the claim is true. After relabelling, we may assume $y_{N,1}$ is close to π .

The composite deformations corresponding to

$$W(\mathbb{F}_{p^f})[[U]] \simeq R^{ord, Y_N \cup \{q_1\} - new} \rightarrow R_{2^*}^{ord, Y_N \cup \{q_1\} - new} \rightarrow W(\mathbb{F}_{p^f})[[U]]/(p^N, g_\pi(U), U^{Ne})$$

and

$$W(\mathbb{F}_{p^f})[[U]] \simeq R^{ord, Y_N - new} \rightarrow R_{2^*}^{ord, Y_N - new} \rightarrow W(\mathbb{F}_{p^f})[[U]]/(p^N, g_\pi(U), U^{Ne})$$

are the same as the latter is nice at q_1 . We know $U \mapsto \pi$ in the latter to give ρ_n so sending U to π in the former gives ρ_n as well. As $y_{N,1}$ is close enough to π , Krasner's lemma implies $W(\mathbb{F}_{p^f})[\frac{1}{p}](\pi) \subset W(\mathbb{F}_{p^f})[\frac{1}{p}](y_{N,1})$. As $\left[W(\mathbb{F}_{p^f})[\frac{1}{p}](y_{N,1}) : W(\mathbb{F}_{p^f})[\frac{1}{p}] \right] \leq \deg(v_{2^*,N}(U)) = e$, the fields $W(\mathbb{F}_{p^f})[\frac{1}{p}](y_{N,1})$ and $W(\mathbb{F}_{p^f})[\frac{1}{p}](\pi)$ are equal. Recall that C is smaller than both $|\pi^n|$ and the Krasner bound on the roots of $g_\pi(U)$. We chose N large enough so that $|y_{N,1} - \pi| < C < |\pi^n|$, so sending U to $y_{N,1}$ in the former sequence gives ρ_n as well. As $R_{2^*}^{ord, Y_N \cup \{q_1\} - new} \simeq W(\mathbb{F}_{p^f})[[U]]/(v_{2^*,N}(U))$, we see ρ_n lifts to an \mathcal{O} -valued weight 2 Galois representation. This proves Theorem 20 in the case where we assumed that $\dim H_{\mathcal{L}}^1(G_{Y_N \cup \{q_1\}}, Ad\bar{\rho}) = 1$ which implied $R^{ord, Y_N \cup \{q_1\} - new} \simeq W(\mathbb{F}_{p^f})[[U]]$. In this case we set $T_2 = T \cup \{q_1\}$.

5.5.2. Case 2. We now deal with the second more involved case. Had we allowed ourselves standard ' $R = T$ ' theorems, then we could assume $R_2^{Y_N \cup \{q_1\}}$ is a finite flat complete intersection and proved the second half of Theorem 20,

$$R^{ord, Y_N \cup \{q_1\} - new} \rightarrow \mathcal{O} \rightarrow \mathcal{O}/(\pi^n)$$

exactly as in case 1). The arbitrary weight ordinary ring might still be a quotient of a power series ring in two variables.

We have:

- $\dim H_{\mathcal{L}}^1(G_{Y_N \cup \{q_1\}}, Ad\bar{\rho}) = 2$, a basis for this space is $\{f, h\}$ where $h \in H^1(G_{Y \cup \{q_1\}}, Ad\bar{\rho}) \setminus H^1(G_{Y_N}, Ad^0\bar{\rho})$ and is ramified at q_1 ,
- $\dim H_{\mathcal{L}^\perp}^1(G_{Y_N \cup \{q_1\}}, Ad\bar{\rho}^*) = 1$ and a basis for this space is $\{\psi\}$ and ψ is ramified at q_1 ,
- $\dim H_{\mathcal{L}}^1(G_{Y_N \cup \{q_1\}}, Ad^0\bar{\rho}) = 1$ and a basis for this space is $\{f\}$,
- $\dim H_{\mathcal{L}^\perp}^1(G_{Y_N \cup \{q_1\}}, Ad^0\bar{\rho}^*) = 1$ and a basis for this space is $\{\tilde{\phi}\}$ where $\{\phi, \tilde{\phi}\}$ is independent and $\tilde{\phi}$ is ramified at q_1 . Recall $\{\phi\}$ formed a basis of $H_{\mathcal{L}^\perp}^1(G_{Y_N}, Ad\bar{\rho}^*)$ and $\phi|_{G_{q_1}} \neq 0$ implies $\phi|_{G_{q_1}} \notin \mathcal{L}_{q_1}^\perp$.

We chose q_1 to satisfy Proposition 24. Now choose a second prime q_2 such that

- $h_{Ad^0 \bar{\rho}} \neq 0$ and $h_{sc} = 0$ so $h \mid_{G_{q_2}} \notin \tilde{\mathcal{L}}_{q_2}$,
- $\psi, \tilde{\phi} \mid_{G_{q_2}} \neq 0$,
- $H^1(G_{Y_N}, Ad^0 \bar{\rho}^*) \mid_{G_{q_2}} = 0$,
- $H^1(G_{Y_N}, Ad^0 \bar{\rho}) \mid_{G_{q_2}} = 0$,
- q_2 is nice for $\rho_R : G_{Y_N} \rightarrow GL_2(W(\mathbb{F}_{p^f})[[U]]) \rightarrow GL_2(W(\mathbb{F}_{p^f})[[U]]/(p^N, U^{Ne}))$.

We have already remarked that the set $\{\phi, \tilde{\phi}\}$ is independent. So is $\{\psi, \phi\}$ as ψ is ramified at q_1 but ϕ is not. If ψ is not in the span of $\{\phi, \tilde{\phi}\}$, then the second and third bullet points are independent Chebotarev conditions. If ψ is in the span of $\{\phi, \tilde{\phi}\}$, then choosing q_2 so that $\phi \mid_{G_{q_2}} = 0$ and $\tilde{\phi} \mid_{G_{q_2}} \neq 0$ implies $\psi \mid_{G_{q_2}} \neq 0$. The above bullet points give independent Chebotarev conditions.

Set $T_2 = Y_N \cup \{q_1, q_2\}$. Then, by our choice of q_2 and part 2) of Proposition 17

- $\dim H_{\tilde{\mathcal{L}}}^1(G_{T_2}, Ad \bar{\rho}) = 1$ and a basis for this space is $\{f\}$,
- $\dim H_{\tilde{\mathcal{L}}^\perp}^1(G_{T_2}, Ad \bar{\rho}^*) = 0$,
- $\dim H_{\mathcal{L}}^1(G_{T_2}, Ad^0 \bar{\rho}) = 1$ and a basis for this space is $\{f\}$,
- $\dim H_{\mathcal{L}}^1(G_{T_2}, Ad^0 \bar{\rho}^*) = 1$ and a basis for this space is $\{\tilde{\phi}\}$.

We have that $R^{ord, T_2 - new} \simeq W(\mathbb{F}_{p^f})[[U]]$ and again by [4] this is a Hida family. Its weight 2 quotient is the middle term of

$$(5.5) \quad W(\mathbb{F}_{p^f})[[U]] = R^{ord, T_2 - new} \twoheadrightarrow R_{2*}^{ord, T_2 - new} = W(\mathbb{F}_{p^f})[[U]]/(m_{2*, N}(U)) \twoheadrightarrow W(\mathbb{F}_{p^f})[[U]]/(p^N, g_\pi(U), U^{Ne})$$

where $m_{2*, N}(U) \in (p^N, g_\pi(U), U^{Ne})$ is a distinguished polynomial of degree at least e . Thus

$$R_{2*}^{ord, T_2 - new} = W(\mathbb{F}_{p^f})[[U]]/(m_N(U)) \twoheadrightarrow \mathbb{F}_{p^f}[[U]]/(U^e)$$

as both q_1 and q_2 were chosen to be nice for this representation to $GL_2(\mathbb{F}_{p^f}[[U]]/(U^e))$, which is unramified outside Y_N . We prove by contradiction that $\deg(m_N(U)) = e$ by contradiction. Suppose $\deg(m_{2*, N}(U)) > e$. Then we have deformations

$$\alpha : G_{T_2} \rightarrow GL_2(\mathbb{F}_{p^f}[[U]]/(U^{e+1})), \quad \beta : G_{Y_N} \rightarrow GL_2(\mathbb{F}_{p^f}[[U]]/(U^{e+1}))$$

where $\alpha = \beta \bmod U^e$ so they differ by a 1-cohomology class. We write this class as $xh_0 + yh_{q_1} + zh_{q_2}$ where $h_0 \in H^1(G_{Y_N}, Ad^0 \bar{\rho})$, $h_{q_i} \in H^1(G_{Y_N \cup \{q_i\}}, Ad^0 \bar{\rho})$ for $i = 1, 2$. We assume h_{q_i} is ramified at q_i for $i = 1, 2$. Note $\alpha, \beta \mid_{G_v} \in \mathcal{C}_v$ for all $v \in Y_N$.

Since $\phi \mid_{G_{q_1}} \neq 0$, $\tilde{\phi} \mid_{G_{q_2}} \neq 0$ parts 1) and 2) of Proposition 16 imply the maps

$$(5.6) \quad H^1(G_{Y_N \cup \{q_1\}}, Ad^0 \bar{\rho}) \rightarrow \oplus_{v \in Y_N} \frac{H^1(G_v, Ad^0 \bar{\rho})}{\mathcal{L}_v}$$

and

$$H^1(G_{Y_N \cup \{q_1, q_2\}}, Ad^0 \bar{\rho}) \rightarrow \oplus_{v \in Y_N \cup \{q_1\}} \frac{H^1(G_v, Ad^0 \bar{\rho})}{\mathcal{L}_v}$$

have images one dimension larger than

$$(5.7) \quad H^1(G_{Y_N}, Ad^0 \bar{\rho}) \rightarrow \oplus_{v \in Y_N} \frac{H^1(G_v, Ad^0 \bar{\rho})}{\mathcal{L}_v}$$

and

$$H^1(G_{Y_N \cup \{q_1\}}, Ad^0 \bar{\rho}) \rightarrow \oplus_{v \in Y_N \cup \{q_1\}} \frac{H^1(G_v, Ad^0 \bar{\rho})}{\mathcal{L}_v}$$

respectively. So successively allowing ramification at q_1 and then q_2 makes the Selmer maps more surjective by one dimension at each stage. But as $H^1(G_{Y_N}, Ad^0 \bar{\rho}^*)|_{G_{q_2}} = 0$, part 5) of Proposition 16 again gives that the map

$$(5.8) \quad H^1(G_{Y_N \cup \{q_2\}}, Ad^0 \bar{\rho}) \rightarrow \bigoplus_{v \in Y_N} \frac{H^1(G_v, Ad^0 \bar{\rho})}{\mathcal{L}_v}$$

has the same image as

$$(5.9) \quad H^1(G_{Y_N}, Ad^0 \bar{\rho}) \rightarrow \bigoplus_{v \in Y_N} \frac{H^1(G_v, Ad^0 \bar{\rho})}{\mathcal{L}_v}.$$

If $h_{q_2}|_{G_{q_2}} \in \mathcal{L}_{q_2}$, then allowing ramification at q_1 would have to add 2 dimensions of surjectivity to the map

$$H^1(G_{Y_N \cup \{q_1, q_2\}}, Ad^0 \bar{\rho}) \rightarrow \bigoplus_{v \in Y_N \cup \{q_1, q_2\}} \frac{H^1(G_v, Ad^0 \bar{\rho})}{\mathcal{L}_v}$$

compared to the map

$$H^1(G_{Y_N \cup q_2}, Ad^0 \bar{\rho}) \rightarrow \bigoplus_{v \in Y_N \cup q_2} \frac{H^1(G_v, Ad^0 \bar{\rho})}{\mathcal{L}_v}.$$

This is impossible by part 1) of Proposition 16 so $h_{q_2}|_{G_{q_2}} \notin \mathcal{L}_{q_2}$. (This idea was used already in [10]).

So when we move from β to α in the equation

$$\alpha = (I + (xh + yh_{q_1} + zh_{q_2})U^e) \beta,$$

what happens at $v \in Y_N$? The image of h_{q_2} under (5.8) was already in the image of (5.9). From (5.7), the image of h_{q_1} lies outside this common image of (5.8) and (5.9). Thus if $y \neq 0$ then for some $v_0 \in Y_N$ we have $(xh + yh_{q_1} + zh_{q_2})|_{G_{v_0}} \notin \mathcal{L}_{v_0}$ so either $\alpha|_{G_{v_0}} \notin \mathcal{C}_{v_0}$ or $\beta|_{G_{v_0}} \notin \mathcal{C}_{v_0}$, a contradiction. Thus $y = 0$.

Now we consider properties at G_{q_2} . By the choice of q_2 , both $\alpha, \beta|_{G_{q_2}} \in \mathcal{C}_{q_2}$ so $(xh_0 + zh_{q_2})|_{G_{q_2}} \in \mathcal{L}_{q_2}$. But the choice of q_2 forces $h_0|_{G_{q_2}} = 0$ and we showed that $h_{q_2} \notin \mathcal{L}_{q_2}$ so $z = 0$.

Finally, we look locally at q_1 . Recall $\beta(Fr_{q_1}) = \begin{pmatrix} q_1(1 + U^e) & 0 \\ 0 & (1 + U^e)^{-1} \end{pmatrix}$ so $\beta \notin \mathcal{C}_{q_1}$. But $h_0|_{G_{q_1}} = 0$ so h_0 cannot move β to \mathcal{C}_{q_1} . We have a contradiction so $m_N(U)$ has degree e . From (5.5) $m_N(U) \in (p^N, g_\pi(U), U^{Ne})$. Now simply proceed as in the Case 1 with N suitably large. \square

5.6. Further corollaries.

Corollary 25. *Let $\bar{\rho} : G_{\mathbb{Q}} \rightarrow GL_2(\mathbb{F}_{p^f})$ be odd, full, ordinary, weight 2 and have determinant ϵ . Let \mathcal{O} be any totally ramified extension of $W(\mathbb{F}_{p^f})$. There exists a set of primes $Y \supset S_0$ such that*

$$W(\mathbb{F}_{p^f})[[U]] \simeq R^{ord, Y} \twoheadrightarrow R^{ord, Y-new}_* \simeq W(\mathbb{F}_{p^f})[[U]]/(h(U)) \simeq \mathcal{O}.$$

The degree of the map to weight space along the Hida family $R^{ord, Y-new}$ is $[\mathcal{O} : W(\mathbb{F}_{p^f})]$ when $\bar{\rho}$ is as in cases 1), 2) and 4) of Proposition 6. In the other cases, the degree is strictly greater than $[\mathcal{O} : W(\mathbb{F}_{p^f})]$. There exists a weight 2 form associated to $\bar{\rho}$ whose completed field of Fourier coefficients has ring of integers \mathcal{O} .

Proof. Use [14] to get a nontrivial weight 2 deformation of $\bar{\rho}$ to

$$W(\mathbb{F}_{p^f})[[U]]/(p, U^2) \simeq \mathcal{O}/(\pi^2),$$

that is the corresponding cohomology class in this deformation to the dual numbers is nonzero. Now apply Theorem 20. \square

Corollary 26. *Let $\bar{\rho} : G_{\mathbb{Q}} \rightarrow GL_2(\mathbb{F}_{p^f})$ be odd, full, ordinary, weight 2 and have determinant ϵ . Let $g(U) \in W(\mathbb{F}_{p^f})[U]$ be a distinguished polynomial of degree e with distinct roots and let $\epsilon > 0$ be given. Then there exists a set $Y \supset S_0$ such that*

$$W(\mathbb{F}_{p^f})[[U]] \simeq R^{ord, Y-new} \twoheadrightarrow R_{2*}^{ord, Y-new} \simeq W(\mathbb{F}_{p^f})[[U]]/(w_{2*}(U))$$

where $w_{2*}(U)$ has degree e and each root of $w_{2*}(U)$ is within ϵ of a root of $g(U)$. Furthermore if $g(U) = \prod g_i(U)$ where $g_i(U)$ is irreducible over $W(\mathbb{F}_{p^f})$ of degree e_i , then $w_{2*}(U) = \prod w_{2*,i}(U)$ where $w_{2*,i}(U)$ is irreducible over $W(\mathbb{F}_{p^f})$ of degree e_i and its roots are within ϵ of the roots of $g_i(U)$.

Proof. First choose ϵ to be less than half the distance between any pair of roots of $g(U)$. Use [14] to get a nontrivial weight 2 deformation of $\bar{\rho}$ to $W(\mathbb{F}_{p^f})[[U]]/(p, U^2)$. Now proceed as in Proposition 22 to get a weight 2 deformation ring surjecting onto $W(\mathbb{F}_{p^f})[[U]]/(p^N, U^{Ne})$ with one dimensional tangent space. Then add more primes so that the ordinary ring is $W(\mathbb{F}_{p^f})[[U]]$ and its weight 2 quotient is $W(\mathbb{F}_{p^f})[[U]]/(w_{2*}(U))$ where $w_{2*}(U) \in (p^N, g(U), U^{Ne})$ is degree e . The argument in case 1) of Theorem 20 implies that for N large enough, each root of $w_{2*}(U)$ is within ϵ of a distinct root of $g(U)$. By the choice of ϵ the roots of $w_{2*}(U)$ are distinct. As $g_i(U)$ is a degree e_i irreducible factor of $g(U)$, let $\{r_{i1}, \dots, r_{ie_i}\}$ be its roots and we know $|r_{ij} - s_{ij}| < \epsilon$ where s_{ij} is a root of $w_{2*}(U)$. We write $r_{ij} = s_{ij} + x_{ij}$. Let σ be an automorphism taking r_{ij} to r_{ik} . Then

$$r_{ik} = \sigma(r_{ij}) = \sigma(s_{ij} + x_{ij}) = \sigma(s_{ij}) + \sigma(x_{ij}).$$

As σ preserves sizes,

$$|r_{ik} - \sigma(s_{ij})| = |\sigma(x_{ij})| = |x_{ij}| < \epsilon,$$

so $\sigma(s_{ij})$ is the root of $w_{2*}(U) \in W(\mathbb{F}_{p^f})[U]$ close to r_{ik} . Thus $\sigma(s_{ij}) = s_{ik}$ and the roots of $w_{2*}(U)$ break up into Galois orbits corresponding to the Galois orbits of the roots of $g(U)$ that are close to them. This proves the factorisation statement. \square

Remarks:

1. We have the following lemma whose proof we owe to N. Fakhruddin. This implies that Corollary 26 gives examples of deformation rings which are non-integrally closed orders in valuation rings. Ralph Greenberg had asked one of us if there were such examples.

Lemma 27. *Let f be a monic polynomial in $\mathbb{Z}_p[X]$ with distinct roots. Then for all monic polynomials $g \in \mathbb{Z}_p[X]$ of degree n which are close enough to f , we have an isomorphism of \mathbb{Z}_p -algebras*

$$\mathbb{Z}_p[X]/(f) \simeq \mathbb{Z}_p[X]/(g).$$

Proof. Let n be the degree of f , and consider the map $\alpha : \mathbb{Z}_p^n \rightarrow \mathbb{Z}_p^n$ defined as follows. Given $\gamma \in \mathbb{Z}_p^n$ we regard it as an element of $\mathbb{Z}_p[X]/(f)$, and send it to the tuple (a_1, \dots, a_n) in \mathbb{Z}_p^n , where the characteristic polynomial of the endomorphism of $\mathbb{Z}_p[X]/(f)$ given by multiplication by γ , is $x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n$. The image of x under α is given by the coefficients of f . Using that f has distinct roots, we see that α is an open mapping in a neighborhood of x , and deduce that all elements in a sufficiently small neighborhood V of $\alpha(x)$ are in $\alpha(U)$ where U is the open neighborhood of $x \in \mathbb{Z}_p^n$ consisting of elements that are congruent to $x \pmod{p}$. We may assume that elements of V correspond to monic polynomials of degree n in $\mathbb{Z}_p[X]$ with distinct roots. From this we deduce that for g close enough to f , we get a monomorphism $\mathbb{Z}_p[X]/(g) \rightarrow \mathbb{Z}_p[X]/(f)$ of \mathbb{Z}_p -algebras with finite cokernel, and further deduce this is an isomorphism by reducing mod p . \square

One could ask, like in Krasner's lemma, for quantitative refinements of this lemma.

2. Let $a \in W(\mathbb{F}_{p^f})$ be a nonsquare in $W(\mathbb{F}_{p^f})$. Then if one chooses $g(U) = U^2 - ap^2$ in Corollary 26, the deformation ring will be an *order* in the unramified degree 2 extension of $W(\mathbb{F}_{p^f})$. Thus one can also obtain nontrivial unramified extensions as completed fields of Fourier coefficients of the modular form corresponding to our Galois representation.

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APPENDIX: MODULARITY OF GEOMETRIC LIFTS ρ VIA p -ADIC APPROXIMATION

We apply Theorem 11 to proving modularity of certain representations $\rho : G_{\mathbb{Q}} \rightarrow GL_2(\mathcal{O})$ where \mathcal{O} is the valuation ring of a finite extension K of \mathbb{Q}_p and $p > 2$.

Theorem 28. *Let $\rho : G_{\mathbb{Q}} \rightarrow GL_2(\mathcal{O})$ be odd, ordinary, full, balanced, ramified at only finitely many primes, weight 2 with determinant ϵ and have modular reduction. Further assume that when $\bar{\rho}$ is split at p , ρ is flat at p . Then ρ is modular.*

The method of proof extends the method of [7] which dealt with the case when K is unramified over \mathbb{Q}_p . Of course these results are contained in those of the various ‘ $R = T$ ’ theorems pioneered by Wiles and Taylor-Wiles. Our point here is to provide a different argument using p -adic approximation. In this appendix the proofs are merely sketched: we are rederiving known results using Theorem 11 and the strategy of [7].

Using Theorem 11, we first prove that for each n , $\rho_n = \rho \bmod (\pi^n)$, is modular of a level which depends on ρ_n . We then lower the level of ρ_n which paves the way to proving modularity of $\rho : G_{\mathbb{Q}} \rightarrow GL_2(\mathcal{O})$ by successive approximation.

We have the following corollary of Theorem 11. We keep the notation of the previous sections: for instance S_0 is the set of primes at which $\bar{\rho}$ is ramified, and S is the set of primes at which ρ is ramified.

Corollary 29. *The representation ρ_n is isomorphic as $\mathcal{O}[G_{\mathbb{Q}}]$ -representation to a submodule of the ordinary part of the p -divisible group associated to J_T tensored over \mathbb{Z}_p with \mathcal{O} . Here J_T is the Jacobian of the projective modular curve $\Gamma_1(Np^r) \cap \Gamma_0(Q_n)$; N is some fixed integer independent of n ; r is an integer which a priori depends on n ; and Q_n is the product of primes in the finite set $T \setminus S$ which depends on n .*

We say that ρ_n as in the corollary arises from J_T . We also say that ρ_n arises from the p -divisible group associated to the ordinary factor J_T^{ord} of J_T .

Proof. This follows from Theorem 11, level raising results of [4] and Hida's theory. These results yield that $R^{ord, T-new}$ surjects onto a T -new ordinary Hecke algebra $\mathbb{T}^{ord, T-new}$ which is finite and torsion free over $\Lambda = \mathbb{Z}_p[[T]]$. But using that $R^{ord, T-new} \simeq W(\mathbb{F}_{p^f})[[U]]$ we deduce that we have an isomorphism of $R^{ord, T-new}$ with $\mathbb{T}^{ord, T-new}$. This implies the corollary by standard arguments. \square

Proposition 30 follows from arguments in [7] with the twist that as we allow primes $q \in T \setminus S_0$ that are $-1 \pmod p$, we have to keep track of the Atkin-Lehner operators W_q for $q \in Q_n = T \setminus S_0$. We have the relation $W_q^2 = \langle q \rangle$.

Proposition 30. *The representation ρ_n arises from the Q_n -old subvariety of J_T^{ord} , and furthermore all the Hecke operators T_r , for r a prime not in T , act on ρ_n by $\text{Trace}(\rho_n(\text{Frob}_r))$.*

Proof. This is an application of Mazur's principle (see §8 of [16]) and uses that $q \not\equiv 1 \pmod p$. The principle relies on the fact that the Frob_q -action on unramified finite $G_{\mathbb{Q}_q}$ -submodules of the torsion points of J_T whose reduction mod q is in the 'toric part' of the reduction mod q of J_T is constrained. Namely, on the 'toric part' the Frobenius Frob_q acts by $-qW_q$ where W_q is the Atkin-Lehner involution. We flesh this out this below.

Consider a prime $q \in Q_n$ where $Q_n = T \setminus S$. Then decompose $\rho_n|_{D_q}$ (which is unramified by hypothesis) into $\mathcal{O}/(\pi^n) \oplus \mathcal{O}/(\pi^n)$, with basis $\{e_n, f_n\}$ with $\text{Frob}_q(e_n) = -W_q \cdot e_n$ and $\text{Frob}_q(f_n) = -qW_q \cdot f_n$ for some character ϵ as above. The action of W_q is by a scalar α_q , and so we have $\text{Frob}_q(e_n) = -\alpha_q e_n$ and $\text{Frob}_q(f_n) = -q\alpha_q f_n$.

Using irreducibility of ρ , Burnside's lemma gives that $\rho(\mathbb{F}_{p^f}[G_{\mathbb{Q}}]) = M_2(\mathbb{F}_{p^f})$ and hence by Nakayama's lemma $\rho_n(\mathcal{O}[G_{\mathbb{Q}}]) = M_2(\mathcal{O}/(\pi^n))$. Thus using the surjection from the Hecke algebra acting on J_T^{ord} to $\mathcal{O}/(\pi^n)$ we deduce that ρ_n arises from an eponymous submodule of $J := J_T^{ord}$.

The fact that the q -old subvariety is stable under the Galois and Hecke action will allow us to deduce that ρ_n arises from the q -old subvariety of J if we can show that e_n is contained in the q -old subvariety of J .

Let \mathcal{J} be the Néron model at q of J . Note that as ρ_n is unramified at q it maps injectively to $\mathcal{J}_{/\mathbb{F}_p}(\bar{\mathbb{F}}_p)$ under the reduction map. Now if the claim were false, as the group of connected components of \mathcal{J} is Eisenstein, we would deduce that the reduction of e_n in $\mathcal{J}^0(\bar{\mathbb{F}}_p)$ maps non-trivially (and hence its image has order divisible by p) to the $\bar{\mathbb{F}}_p$ -points of the torus which is the quotient of \mathcal{J}^0 by the image of the q -old subvariety (in characteristic q). But as we recalled above, it is well-known (see §8 of [16]) that Frob_q acts on the $\bar{\mathbb{F}}_p$ -valued points of this toric quotient (isogenous to the torus T of \mathcal{J}^0 , the latter being a semiabelian variety that is an extension of an abelian variety by T) by $-qW_q$ which gives a contradiction. Now taking another prime $q' \in Q_n$ and working within the q -old subvariety of J , by the same argument we see that ρ_n occurs in the $\{q, q'\}$ -old subvariety of J , and eventually that ρ_n occurs in the Q_n -old subvariety of J . The last part of the proposition is then clear. \square

We finish the proof of the main theorem of the appendix Theorem 28.

Proof. For an integer N prime to p , denote by $J_1(Np^\infty)^{ord}$ the direct limit of the ordinary parts of $J_1(Np^r)$ as r varies. From Proposition 30 it is easy to deduce that ρ_n , the mod (π^n) reduction of ρ , arises from $J_1(Np^\infty)^{ord}$ for some fixed integer N that is independent of n . Let \mathbb{T} be the Hida Hecke algebra acting on $J_1(Np^\infty)^{ord}$, generated by the Hecke operators T_r with r prime and prime to Np . We claim that the ρ_n give compatible morphisms from \mathbb{T} to the $\mathcal{O}/(\pi^n)$. To get these morphisms, let V_n denote a realisation of the representation ρ_n in $J_1(Np^\infty)^{ord}$ which exists by Proposition 30. Then V_n is $G_{\mathbb{Q}}$ -stable, and hence \mathbb{T} -stable (because of the Eichler-Shimura congruence relation mod r , that gives an equality of correspondences $T_r = \text{Frob}_r + r\langle r \rangle \text{Frob}_r^{-1}$ where Frob_r is the Frobenius

morphism at r). So V_n is a \mathbb{T} -module, and because of the absolute irreducibility (only the scalars commute with the $G_{\mathbb{Q}}$ -action) \mathbb{T} acts via a morphism $\alpha_n : \mathbb{T} \rightarrow \mathcal{O}/(\pi^n)$ as desired, and the α_n 's are compatible again because of the congruence relation. This gives a morphism $\alpha : \mathbb{T} \rightarrow \mathcal{O}$ such that the representation associated to α is isomorphic to ρ which finishes proof of the theorem. Then using that the determinant of ρ is ϵ and Hida's control theorem, we deduce that ρ arises from a weight 2 newform. \square

Improvements to the method. The weight 2 assumption on $\bar{\rho}$ and the lifts we consider (and the assumption on the determinant) is for convenience, and our methods apply to ρ of weight $k \geq 2$ (the Hodge-Tate weights are $(k-1, 0)$), provided that $\bar{\rho}$ is distinguished at p .

The *fullness* assumption on ρ and ρ_n used in the proof of Theorem 11 arises from the fact that in its absence Lemma 14 is not true. On the other hand one can prove a more qualitative but less restrictive version of this lemma.

Lemma 31. *Let $p \geq 3$. Recall \mathcal{O} is the ring of integers of a finite extension of \mathbb{Q}_p , with uniformiser π and residue field \mathbb{F}_{p^f} . Let $G \subset GL_2(\mathcal{O})$ be a closed subgroup. Assume that the image G_1 of $G \rightarrow GL_2(\mathbb{F}_{p^f})$, contains $SL_2(\mathbb{F}_{p^f})$ and satisfies hypothesis of Lemma 13. Then $\dim H^1(G, Ad^0 \bar{\rho})$ is a finite abelian group.*

The proof uses (i) $H^1(G_1, Ad^0 \bar{\rho}) = 0$ (cf. Lemma 13), and (ii) the kernel of the homomorphism $G \rightarrow G_1$ is a finitely generated pro- p group.

G. Böckle has observed that using such a lemma, one can remove the assumption on fullness of image of ρ , made in the arguments in the appendix, by using base change to totally real solvable extensions F/\mathbb{Q} and considering nearly ordinary deformations of $\bar{\rho}|_{G_F}$ and Hida's nearly ordinary Hecke algebras. Choose a totally real solvable extension F/\mathbb{Q} disjoint from the field cut out by ρ , and whose degree $d = [F : \mathbb{Q}]$ is $> \dim H^1(G, Ad^0 \bar{\rho})$. Then by choice of F and the technique of killing dual Selmer groups, and obtaining smooth quotients of deformation rings of the expected dimension of this paper, one obtains nearly ordinary deformation rings that are power series rings in $d + \delta + 1$ variables, where d is the degree of F over \mathbb{Q} and δ the Leopoldt defect for F and p , and such that $\rho \bmod \pi^n$ arises from the corresponding universal deformation. One then would exploit the fact that Hida's nearly ordinary Hecke algebra is finite flat over $\mathbb{Z}_p[[X_0, \dots, X_{d+\delta}]]$. By more elaborate level lowering methods (as in [19]) one would then by a similar strategy as above prove that $\rho|_{G_F}$ is automorphic which suffices as F/\mathbb{Q} is solvable.

To make the present method of modularity lifting more robust, one would ultimately hope to also remove the conditions of *ordinarity* and being *balanced* on geometric ρ , and show assuming $\bar{\rho}$ is modular and irreducible, that $\rho \bmod (\pi^n)$ is modular of some level for each n , and hence by level lowering techniques deduce that ρ itself arises for a new form. For this again base change to solvable totally real extension of \mathbb{Q} and the *trivial primes* of [6] might be useful to remove the *balanced condition*, and to remove ordinarity one would have to use Coleman families and work on the eigenvariety.

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